MATHEMATICS

SECTION 1 (Maximum Marks : 28)

- This section contains **SEVEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four options is (are) correct
- For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks	: +4	If only the bubble(s) corresponding to all the correct option(s) is(are) darkened
Partial Marks	: +1	For darkening a bubble corresponding **to each correct option,** provided NO incorrect option is darkened
Zero Marks	: 0	If none of the bubbles is darkened
Negative Marks	: –2	In all other cases

- For example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will get +4 marks; darkening only (A) and (D) will get +2 marks; and darkening (A) and (B) will get –2 marks, as a wrong option is also darkened.

37. Let a,b,x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\mathrm{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is (are) possible value(s) of x?

[A] $-1+\sqrt{1-y^2}$ [B] $1-\sqrt{1+y^2}$ [C] $1+\sqrt{1+y^2}$ [D] $-1-\sqrt{1-y^2}$

Ans. **(A, D)**

Sol. $$\frac{\frac{az+b}{z+1} - \frac{a\bar{z}+b}{\bar{z}+1}}{2i} = y$$

$$\Rightarrow \frac{a|z|^2 + b\bar{z} + az + b - a|z|^2 - bz - a\bar{z} - b}{|z+1|^2} = 2iy$$

$$\Rightarrow \frac{(a-b)(z-\bar{z})}{|z+1|^2} = 2iy$$

$$\Rightarrow |z+1|^2 = 1 \qquad (\because a-b=1 \text{ and } z-\bar{z} = 2iy, y \neq 0)$$

$$\Rightarrow (x+1)^2 + y^2 = 1$$

$$\Rightarrow (x+1)^2 = 1 - y^2 \qquad \Rightarrow (x+1) = \pm\sqrt{1-y^2}$$

$$\therefore x = -1 \pm \sqrt{1-y^2}$$

$$= -1+\sqrt{1-y^2}, -1-\sqrt{1-y^2}$$

38. Let $f : R \to (0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval $(0,1)$?

[A] $f(x)+\int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

[B] $x^9 - f(x)$

[C] $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

[D] $e^x - \int_0^x f(t)\sin t\,dt$

Ans. **(B, C)**

Sol. **[A]** Let $G_1(x) = f(x)+\int_0^{\pi/2} f(t)\sin t\,dt > 0 \;\forall x \in (0,1)$

[B] Let $G_2(x) = x^9 - f(x)$

$G_2(0) = -f(0) < 0$

$G_2(1) = 1 - f(1) > 0$

$\therefore$ $G_2(x)$ has the value zero at some point in the interval $(0,1)$

[C] Let $G_3(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

$G_3(0) = -\int_0^{\frac{\pi}{2}} f(t)\cos t\,dt < 0$

$G_3(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t)\cos t\,dt > 0$

$\therefore$ $G_3(x)$ has the value zero at some point in the interval $(0,1)$

[D] Let $G_4(x) = e^x - \int_0^x f(t)\sin t\,dt$, $G'_4(x) = e^x - f(x)\sin x > 0 \;\forall x \in (0,1)$

$G_4(0) = 1 > 0$

$\therefore G_4(x) > 0 \;\forall x \in (0,1)$

39. If 2x – y + 1 = 0 is a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{16}=1$, then which of the following CANNOT be sides of a right angled triangle?

[A] 2a, 4, 1 [B] a, 4, 1 [C] a, 4, 2 [D] 2a, 8, 1

Ans. **(B, C, D)**

Sol. $y=2x+1,\ m=2,\ c=1$

$a^2=a^2, b^2=16$

$\because c^2=a^2m^2-b^2=4a^2-16$

$\Rightarrow 4a^2=c^2+16=17.$

[A] $\therefore (2a)^2=(4)^2+(1)^2$

$\therefore$ sides are of right angled triangle.

[B] $a^2=\frac{17}{4},\ 4^2=16,\ 1^2=1$

$\therefore$ sides are not of right angled triangle.

[C] $a^2=\frac{17}{4},\ 4^2=16,\ 2^2=4$

$\therefore$ sides are not of right angled triangle.

[D] $a^2=17,\ 8^2=64,\ 1^2=1$

$\therefore$ sides are not of right angled triangle.

40. Let X and Y be two events such that $P(X)=\frac{1}{3}$, $P(X\,|\,Y)=\frac{1}{2}$ and $P(Y\,|\,X)=\frac{2}{5}$. Then

[A] $P(X\cap Y)=\frac{1}{5}$ [B] $P(Y)=\frac{4}{15}$ [C] $P(X'\,|\,Y)=\frac{1}{2}$ [D] $P(X\cup Y)=\frac{2}{5}$

Ans. **(B, C)**

Sol. $P(X)=\frac{1}{3}$

$P\left(\frac{X}{Y}\right)=\frac{1}{2}$

$P\frac{(X\cap Y)}{P(Y)}=\frac{1}{2}$ ....(1)

$P\left(\frac{Y}{X}\right)=\frac{P(X\cap Y)}{P(X)}=\frac{2}{5}$ ....(2)

From (1) and (2)

$\frac{1}{2}P(Y) = \frac{2}{5}P(X)$

$P(Y) = \frac{4}{15}$ ....(3)

$P(X \cap Y) = \frac{2}{15}$ ....(4)

$P\left(\frac{X'}{Y}\right) = \frac{P(X' \cap Y)}{P(Y)} = \frac{P(Y) - P(X \cap Y)}{P(Y)} = 1 - \frac{1}{2} = \frac{1}{2}$ ....(5)

$P(X \cup Y) = \frac{1}{3} + \frac{4}{15} - \frac{2}{15} = \frac{7}{15}$ ....(6)

41. Which of the following is(are)NOT the square of a 3×3 matrix with real entries?

[A] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ [B] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ [C] $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ [D] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Ans. (A, C)

Sol. Let A be a matrix of real entries with determinant x. |A| = x

$\det\left(A^2\right) = (\det A)(\det A) = x^2$

So, $\det\left(A^2\right)$ must be positive

But det of option A & C are negative so A and C are not possible

42. Let [x] be the greatest integer less than or equal to x. Then at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?

[A] x = 0 [B] x = 1 [C] x = 2 [D] x = –1

Ans. (B,C,D)

Sol. $f(x) = x\cos(\pi x + \pi[x])$

$= (-1)^{[x]} x\cos(\pi x)$

which is discontinuous at every $x \in I - \{0\}$.

43. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation 2x + y = p, and midpoint (h,k), then which of the following is(are) possible value(s) of p,h and k?

[A] $p = 2, h = 3, k = -4$ [B] $p = 5, h = 4, k = -3$

[C] $p = -1, h = 1, k = -3$ [D] $p = -2, h = 2, k = -4$

Ans. (A)

Sol. Given line y + 2x = p is not a tangent to the parabola $y^2 = 16x$

$\Rightarrow p \neq -2$

Also, $y^2 = 8(p - y)$

$y^2 + 8y - 8p = 0$ .....(i)

Given mid points is (h,k)

So, $\frac{y_1 + y_2}{2} = k = -4$

and mid points lies on the line y + 2x = p

So, k + 2h = p

$\therefore k = -4$ and $h = \frac{p+4}{2}$

Which is satisfied by option (A) only.

SECTION 2 (Maximum Marks : 15)

- This section contains **FIVE** questions
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from 0 to 9, both inclusive
- For each question, darken the bubble corresponding to the correct integer in the ORS
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct answer is darkened
Zero Marks : 0 In all other cases

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44. The sides of a right angled triangled are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side?

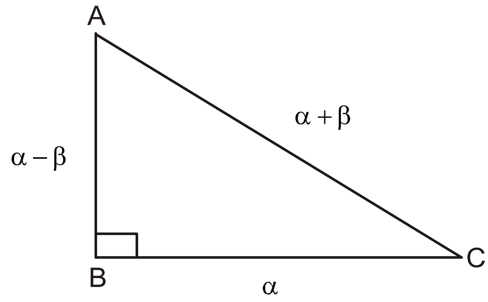
Ans. **(6)**

Sol.

Given

$\frac{1}{2}\alpha(\alpha - \beta) = 24$

$\therefore \alpha(\alpha - \beta) = 48$ ....(i)



For right angle triangle

$(\alpha + \beta)^2 = \alpha^2 + (\alpha - \beta)^2$

$\Rightarrow \alpha = 4\beta$ $(\because \alpha \neq 0)$ ....(ii)

From (i) & (ii)

$\alpha = 8$ & $\beta = 2$

So, sides are 6, 8 & 10

45. For a real number α, if the system $\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ of linear equations, has infinitely many solutions, then $1+\alpha+\alpha^2 =$

Ans. (1)

Sol. If equations has infinitely many solutions.

So, $D = D_x = D_y = D_z = 0$

Where, $D = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} = 0$

$\Rightarrow 1(1-\alpha^2) - \alpha(\alpha-\alpha^3) + \alpha^2(0) = 0 \quad \Rightarrow 1-\alpha^2-\alpha^2+\alpha^4 = 0$

$\Rightarrow (1-\alpha^2)(1-\alpha^2) = 0 \quad \Rightarrow (1-\alpha^2)^2 = 0$

$\Rightarrow 1-\alpha^2 = 0 \qquad \Rightarrow \alpha = \pm 1$

When $\alpha = -1$

$D_x = D_y = D_z = 0$

$\Rightarrow 1+\alpha+\alpha^2 = 1$

46. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I , J . Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letters is repeated twice and no other letter is repeated. Then $\frac{y}{9x} =$

Ans. (5)

Sol. $x = 10!$

$y = {}^{10}C_1 \times {}^{9}C_8 \times \frac{10!}{2!}$

$\therefore \frac{y}{9x} = \frac{10\times 9\times \frac{10!}{2!}}{9\times 10!} = 5$

47. For how many values of p, the circle $x^2+y^2+2x+4y-p=0$ and the coordinates axes have exactly three common points?

Ans. (2)

Sol. Comparing $x^2+y^2+2x+4y-p=0$ with $x^2+y^2+2gx+2fy+c=0$ we get

g = 1, f = 2 and c = –p

For the given condition we must either have

(i) $g^2 - c = 0$ and $f^2 - c > 0$ or (ii) $f^2 - c = 0$ and $g^2 - c > 0$ or

(iii) p = 0

when $g^2 - c = 0$ we have p = – 1 which satisfies $f^2 - c > 0$

and when $f^2 - c = 0$ we have p = – 4 which does not satisfy $g^2 - c > 0$.

And for p = 0 the required condition is satisfied.

48. Let $f : R \to R$ be a differentiable function such that $f(0) = 0, f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$.

If $g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\operatorname{cosec} t - (\cot(t))(\operatorname{cosec} t) f(t)]\,dt$ for $x \in (0, \frac{\pi}{2}]$, then $\lim_{x \to 0} g(x) =$

Ans. (2)

Sol. $g(x) = \int_x^{\pi/2} (f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t))\,dt$

$= \int_x^{\pi/2} \frac{d}{dt}(f(t)\operatorname{cosec} t)\,dt$ **=** $3 - f(x)\operatorname{cosec} x$

Now, $\lim_{x\to 0} g(x) = \lim_{x \to 0}(3 - f(x)\operatorname{cosec} x)$

$= 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$ $= 3 - \lim_{x \to 0}\frac{f'(x)}{\cos x}$

$= 3 - f'(0) = 3 - 1 = 2$

SECTION 3 (Maximum Marks : 18)

- This section contains **SIX** questions of matching type
- This section contains **TWO** tables (each haivng 3 columns and 4 rows)
- Based on each table, there are **THREE** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is correct
- For each question, darken the bubble corresponding to the correct option in the ORS
- For each question, marks will be awarded in one of the following categories :

  Full Marks : +3 If only the bubble corresponding to the correct option is darkened

  Zero Marks : 0 If none of the bubbles is darkened

  Negative Marks : –1 In all other cases

Answer Q. 49, Q.50 and Q.51 by appropriately matching the information given in the three columns of the following table

Column 1, 2 & 3 contain conics, equations of tangents to the conics and points of contact, respectively.

	Column 1		Column 2		Column 3
(I)	$x^2+y^2=a^2$	**(i)**	$my=m^2x+a$	**(P)**	$\left(\frac{a}{m^2},\frac{2a}{m}\right)$
(II)	$x^2+a^2y^2=a^2$	**(ii)**	$y=mx+a\sqrt{m^2+1}$	**(Q)**	$\left(\frac{-ma}{\sqrt{m^2+1}},\frac{a}{\sqrt{m^2+1}}\right)$
(III)	$y^2=4ax$	**(iii)**	$y=mx+\sqrt{a^2m^2-1}$	**(R)**	$\left(\frac{-a^2m}{\sqrt{a^2m^2+1}},\frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV)	$x^2-a^2y^2=a^2$	**(iv)**	$y=mx+\sqrt{a^2m^2+1}$	**(S)**	$\left(\frac{-a^2m}{\sqrt{a^2m^2-1}},\frac{-1}{\sqrt{a^2m^2-1}}\right)$

49. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3},\frac{1}{2}\right)$ is found to be $\sqrt{3}x+2y=4$, then which of the following options is the only CORRECT combination?

(A) (IV) (iii) (S) (B) (II) (iii) (R)

(C) (IV) (iv) (S) (D) (II) (iv) (R)

Ans. **(D)**

Sol. For $x^2+a^2y^2=a^2$ point $\left(\sqrt{3},\frac{1}{2}\right)$ is lying on the curve

$3+\frac{a^2}{4}=a^2 \Rightarrow a^2=4$ i.e. curve : $\frac{x^2}{4}+y^2=1$

Tangent at the point $\left(\sqrt{3},\frac{1}{2}\right)$ is

$$\frac{x\sqrt{3}}{4} + y \cdot \frac{1}{2} = 1 \;\Rightarrow \sqrt{3}x + 2y = 4$$

$\therefore$ Equation of the curve $x^2 + a^2y^2 = a^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{1} = 1$

Equation of tangent : $y = mx + \sqrt{a^2m^2 + 1}$

Point of contact : $\left(\frac{-a^2m}{\sqrt{m^2+1}}, \frac{1}{\sqrt{m^2+1}}\right)$

50. If a tangent to a suitable conic (Column 1) is found to be y = x + 8 and its point of contact is (8, 16), then which of the following options is the only CORRECT combination?

(A) (III) (i) (P)
(B) (I) (ii) (Q)
(C) (II) (iv) (R)
(D) (III) (ii) (Q)

Ans. (A)

Sol. For $y^2 = 4ax$

Equation of tangent : $y = mx + \frac{a}{m} \Rightarrow my = m^2x + a$

Also, point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$\because$ (8, 16) lies on the curve $y^2 = 4ax$

$\therefore 16^2 = 4a \cdot 8 \Rightarrow a = 8$

Equation of tangent y = x + 8

51. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1, 1), then which of the following options is the only CORRECT combination for obtaining its equation?

(A) (II) (ii) (Q)
(B) (I) (i) (P)
(C) (I) (ii) (Q)
(D) (III) (i) (P)

Ans. (C)

Sol. For $a = \sqrt{2}$

Curves are

(I) $x^2 + y^2 = 2$

(II) $x^2 + 2y^2 = 2$

(III) $y^2 = 4\sqrt{2}\,x$

(IV) $x^2 - 2y^2 = 2$

Hence, (–1, 1) satisfies only curve (I)

So, curve is $x^2 + y^2 = a^2$ i.e. a circle

Equation of tangent : $y = mx \pm a\sqrt{1+m^2}$

$\therefore$ Point of contact is $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$

Answer Q. 52, Q.53 and Q.54 by appropriately matching the information given in the three columns of the following table

Let $f(x) = x + \log_e x - x\log_e x,\ x \in (0, \infty)$

* Column 1 contains information about zeros of $f(x), f'(x)$ and $f''(x)$.
* Column 2 contains information about the limiting behavior of $f(x), f'(x)$ and $f''(x)$ at infinity.
* Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

	Column 1		**Column 2**		**Column 3**
(I)	$f(x) = 0$ for some $x \in (1, e^2)$	**(i)**	$\lim_{x\to\infty} f(x) = 0$	**(P)**	f is increasing in (0, 1)
(II)	$f'(x) = 0$ for some $x \in (1, e)$	**(ii)**	$\lim_{x\to\infty} f(x) = -\infty$	**(Q)**	f is decreasing in (e, e^2)
(III)	$f'(x) = 0$ for some $x \in (0, 1)$	**(iii)**	$\lim_{x\to\infty} f'(x) = -\infty$	**(R)**	f' is increasing in (0, 1)
(IV)	$f''(x) = 0$ for some $x \in (1, e)$	**(iv)**	$\lim_{x\to\infty} f''(x) = 0$	**(S)**	f' is decreasing in (e, e^2)

52. Which of the following options is the only CORRECT combination ?

(A) (I) (ii) (R) (B) (IV) (i) (S)
(C) (III) (iv) (P) (D) (II) (iii) (S)

Ans. (D)

53. Which of the following options is the only CORRECT combination ?

(A) (I) (i) (P) (B) (II) (ii) (Q)
(C) (III) (iii) (R) (D) (IV) (iv) (S)

Ans. (B)

54. Which of the following options is the only **INCORRECT** combination ?

(A) (II) (iii) (P) (B) (I) (iii) (P)
(C) (III) (i) (R) (D) (II) (iv) (Q)

Ans. (C)

Sol. for Q. No. 52 - 54

$f(x) = x + \log_e x - x\log_e x$

f(x) is continuous in $(0, \infty)$

$f'(x) = \frac{1}{x} - \log_e x$

$f'(x)$ is continuous in $(0, \infty)$

$f''(x) = \frac{-(1+x)}{x^2}$

$f''(x)$ is continuous in $(0, \infty)$

$$f'''(x) = \frac{2}{x^3} + \frac{1}{x^2}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\log_e x\left(\frac{1}{\log_e x} + \frac{1}{x} - 1\right) = -\infty$$

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{1}{x} - \log_e x\right) = -\infty$$

$$\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} \frac{-(1+x)}{x^2} = 0$$

$f'(x) = \frac{1}{x} - \log_e x > 0 \quad \forall x \in (0,1)$ $\Rightarrow$ f is increasing in (0, 1)

$f'(x) = \frac{1}{x} - \log_e x < 0 \quad \forall x \in (e, e^2)$ $\Rightarrow$ f is decreasing in (e, e^2)

$f''(x) < 0 \quad \forall x \in (0,1) \cup (e, e^2)$ $\Rightarrow$ $f'(x)$ is decreasing in $(0, 1) \cup (e, e^2)$ ....(i)

$f'''(x) > 0 \quad \forall x \in (1, e) \Rightarrow f''(x)$ is increasing in (1, e) ....(ii)

$f(1) = 1 > 0$ ....(iii)

$f(e^2) = 2 - e^2 < 0$ ....(iv)

$f'(1) = 1 > 0$ ....(v)

$f'(e) = \frac{1}{e} - 1 < 0$ ....(vi)

$\lim_{x\to 0^+} f'(x) = \infty > 0$ ....(vii)

$f''(1) = -2 < 0$ ....(viii)

$f''(e) = \frac{-(1+e)}{e^2} < 0$ ....(ix)

From (iii) & (iv): f(x) = 0 for some $x \in (1, e^2)$

From (v) & (vi) : $f'(x) = 0$ for some $x \in (1, e)$

From (i), (v) & (vii)

$f'(x) = 0$ not possible in $x \in (0,1)$

From (ii), (viii) & (ix)

$f''(x) = 0$ not possible in $x \in (1, e)$

Correct Statements :

Column 1	**Column 2**	**Column 3**
(I), (II)	(ii), (iii), (iv)	(P), (Q), (S)