MATHEMATICS

SECTION 1 (Maximum Marks : 21)

- This section contains **SEVEN** questions
- Each question has **FOUR** options (A), (B), (C) and (D), **ONLY ONE** of these four options is correct.
- For each question, darken the bubble corresponding to the correct option in the ORS.
- For each question, marks will be awarded in <u>one of the following categories</u> :

  Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

  Full Marks : 0 If none of the bubbles is darkened.

  Negative Marks : –1 In all other cases.

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37. Let S = {1, 2, 3,...,9}. For k = 1, 2,....5, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$

[A] 125 [B] 210 [C] 252 [D] 126

Ans. **(D)**

Sol. In's odd Nos, are five(s) and even number = 4 (four)

$N_1 = {}^4C_4 \times {}^5C_1 = 5$

$N_2 = {}^4C_3 \times {}^5C_2 = 4 \times 10 = 40$

$N_3 = {}^4C_2 \times {}^5C_3 = 6 \times 10 = 60$

$N_4 = {}^4C_1 \times {}^5C_4 = 4 \times 5 = 20$

$N_5 = {}^4C_0 \times {}^5C_5 = 1$

$\therefore$ $N_1 + N_2 + N_3 + N_4 + N_5 = 5 + 40 + 60 + 20 + 1 = 126$

38. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes 2x + y – 2z = 5 and 3x – 6y – 2z = 7, is

[A] $-14x + 2y + 15z = 3$ [B] $14x - 2y + 15z = 27$

[C] $14x + 2y - 15z = 1$ [D] $14x + 2y + 15z = 31$

Ans. **(D)**

Equation of plane

$(x-1)a + (y-1)b + (z-1)c = 0$ ....(i)

Also, $2a + b - 2c = 0$ and $3a - 6b - 2c = 0 \Rightarrow \frac{a}{14} = \frac{b}{2} = \frac{c}{15}$

$\therefore$ Equation of plane is 14x + 2y + 15z = 31

39. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that $\overrightarrow{OP}\cdot\overrightarrow{OQ}+\overrightarrow{OR}\cdot\overrightarrow{OS}=\overrightarrow{OR}\cdot\overrightarrow{OP}+\overrightarrow{OQ}\cdot\overrightarrow{OS}=\overrightarrow{OQ}\cdot\overrightarrow{OR}+\overrightarrow{OP}\cdot\overrightarrow{OS}$. Then the triangle PQR has S as its

[A] incentre [B] circumcentre [C] orthocentre [D] centroid

Ans. **(C)**

Sol. $(\overrightarrow{OQ}-\overrightarrow{OR})\cdot(\overrightarrow{OP}-\overrightarrow{OS})=0$ ....(i)

$(\overrightarrow{OP}-\overrightarrow{OQ})\cdot(\overrightarrow{OR}-\overrightarrow{OS})=0$ ....(ii)

$\Rightarrow \overrightarrow{QR}\cdot\overrightarrow{PS}=0$ and $\overrightarrow{PQ}\cdot\overrightarrow{RS}=0$

$\therefore$ S is orthocentre of ΔPQR

40. How many 3×3 matrices M with entries from {0, 1, 2} are there, for which the sum of the diagonal entries of M^TM is 5?

[A] 162 [B] 135 [C] 126 [D] 198

Ans. **(D)**

Sol. Let $m=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

sum of diagonal elements of M^TM is $= a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2=5$

Here, any five entries are 1, rest zero

or any seven entries are zero and rest are 1 and 2.

so, ${}^9C_5+{}^9C_7\times 2=198$

41. Three randomly chosen nonnegative integers x,y and z are found to satisfy the equation x + y + z = 10. Then the probability that z is even, is

[A] $\frac{6}{11}$ [B] $\frac{36}{55}$ [C] $\frac{1}{2}$ [D] $\frac{5}{11}$

Ans. **(A)**

Sol. x + y + z = 10

number of non negative solution = ${}^{12}C_2=66$

x + y = 10 – z, when z is even

number of solution = 11 + 9 + 7 + 5 + 3 + 1 = 36

required probability $=\frac{6}{11}$

42. If $f:R\to R$ is a twice differentiable function such that $f''(x)>0$ for all $x\in R$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}, f(1)=1$, then

[A] $\frac{1}{2}<f'(1)\le 1$ [B] $0<f'(1)\le\frac{1}{2}$ [C] $f'(1)\le 0$ [D] $f'(1)>1$

Ans. **(D)**

Sol. since f(x) is continuous and differentiable in $\left(\frac{1}{2},1\right)$

$\Rightarrow f'(x)=\dfrac{f(1)-f\left(\frac{1}{2}\right)}{1-\frac{1}{2}}=1$ for at least one $x\in\left(\frac{1}{2},1\right)$

But $f''(x)>0\Rightarrow f'(x)$ is increasing function

$\Rightarrow f'(1)>f'(x)$ for $x\in\left(\frac{1}{2},1\right)$

$\Rightarrow f'(1)>1$

43. If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx,\ x>0$ and

$y(0)=\sqrt{7}$, then $y(256)=$

[A] 80 [B] 9 [C] 16 [D] 3

Ans. **(D)**

Sol. $dy=\dfrac{dx}{8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)}$

Let $4+\sqrt{9+\sqrt{x}}=t^2$

$\dfrac{dx}{8\sqrt{x}\sqrt{9+\sqrt{x}}}=t\,dt$

$\therefore y=\int dt=\sqrt{4+\sqrt{9+\sqrt{x}}}+C$

$y(0)=\sqrt{7}\Rightarrow C=0$

y(256) = 3

SECTION 2 (Maximum Marks : 28)

- This section contains **SEVEN** questions.
- Each question has **FOUR** options (A), (B), (C) and (D), **ONE OR MORE THAN ONE** of these four options(s) is (are) correct.
- For each question, darken the bubble(s) corresponding to all the correct options(s) in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +4 If only the bubble(s) corresponding to the correct option(s) is(are) darkened.

Partial Marks : +1 For darkening a bubble corresponding **to each correct option,** provided NO incorrect option is darkened

Zero Marks : 0 If none of the bubbles is darkened

Negative Marks : –2 In all other cases

- For example, if (A), (C) and (D) are all the correct options for a question, darkening all these three will result in +4marks; darkening only (A) and (D) will result in +2 marks; and darkening (A) and (B) will result in –2 marks, as a wrong option is also darkened.

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44. If the line x = α divides the area of region $R = \{(x,y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal part, then

[A] $0 < \alpha \le \frac{1}{2}$ [B] $2\alpha^4 - 4\alpha^2 + 1 = 0$ [C] $\alpha^4 + 4\alpha^2 - 1 = 0$ [D] $\frac{1}{2} < \alpha < 1$

Ans. (B, D)

Sol. $\int_0^1 (x - x^3)dx = \frac{1}{4}$

So, $\int_0^\alpha (x - x^3)dx = \frac{1}{8} = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^\alpha$

or $\frac{\alpha^2}{2} - \frac{\alpha^4}{4} = \frac{1}{8}$

or $2\alpha^4 - 4\alpha^2 + 1 = 0,$ $\alpha^2 = 1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}$(not valid)

Clearly $\frac{1}{2} < \alpha < 1.$

45. If $f(x)=\begin{vmatrix}\cos(2x) & \cos(2x) & \sin(2x)\\ -\cos x & \cos x & -\sin x\\ \sin x & \sin x & \cos x\end{vmatrix}$, then

[A] f(x) attains its maximum at x = 0

[B] f(x) attains its minimum at x = 0

[C] $f'(x)=0$ at more than three points in $(-\pi,\pi)$

[D] $f'(x)=0$ at exactly three points in $(-\pi,\pi)$

Ans. (A, C)

Sol. $f(x)=\cos 2x(\cos^2 x+\sin^2 x)-\cos 2x(-\cos^2 x+\sin^2 x)+\sin 2x(-\sin x\cos x-\sin x\cos x)$

$f(x)=\cos 2x+\cos^2 2x-(\sin^2 2x)$

$f(x)=\cos 2x+\cos 4x$

f(x) is maximum at $x=0,\pi$

$f'(x)=-2\sin 2x-4\sin 4x$

$=-2\sin 2x-8\sin 2x\cos 2x$

$f'(x)=-2\sin 2x(1+4\cos 2x)$

$f'(x)=0$ at 7 points is $x\in(-\pi,\pi)$

46. If $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$, then

[A] $I>\frac{49}{50}$

[B] $I<\frac{49}{50}$

[C] $I<\log_e 99$

[D] $I>\log_e 99$

Ans. (A,C)

Sol. $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(k+1)}dx < \sum_{k=1}^{98}\int_k^{k+1}\frac{1}{x}dx = \ln 99$

$I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx > \sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{(x+1)^2}dx > \sum_{k=1}^{98}\frac{1}{k+2} > \sum_{K=1}^{98}\frac{1}{100}=\frac{49}{50}$

47. Let $f(x)=\frac{1-x(1+|1-x|)}{|1-x|}\cos\left(\frac{1}{1-x}\right)$ for $x\neq 1$. Then

[A] $\lim_{x\to 1^+} f(x)=0$

[B] $\lim_{x\to 1^-} f(x)=0$

[C] $\lim_{x\to 1^+} f(x)$ does not exist

[D] $\lim_{x\to 1^-} f(x)$ does not exist

Ans. (B,C)

Sol. $f(x) = \frac{1-x(1+|1-x|)}{|1-x|}\cos\frac{1}{1-x}$ for $x \neq 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{1-2x+x^2}{1-x}\cos\frac{1}{1-x} = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1-x^2}{x-1}\cos\frac{1}{1-x} = \lim_{x\to 1^+} -(1+x)\cos\frac{1}{1-x}$$

which does not exist.

48. If $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and f(0) = 1, then

[A] f(x) is increasing in $(0, \infty)$

[B] $f'(x) < e^{2x}$ in $(0, \infty)$

[C] $f(x) > e^{2x}$ in $(0, \infty)$

[D] $f(x)$ is decreasing in $(0, \infty)$

Ans. (A,C)

Sol. $f'(x) > 2f(x)$

$\Rightarrow e^{-2x} f'(x) - 2e^{-2x} f(x) > 0$ ....(i)

$\frac{d}{dx}\left(e^{-2x} f(x)\right) > 0 \ \forall x \in \mathbb{R}$

Let $g(x) = e^{-2x} f(x)$

for $x > 0$, $g(x) > g(0)$

$e^{-2x} f(x) > 1$

$f(x) > e^{2x} \ \forall x \in (0, \infty)$

Now, from equation (i)

$e^{-2x} f'(x) > 2e^{-2x} f(x)$ $\quad \therefore e^{-2x} f'(x) > 2 \ \forall x \in (0, \infty)$

$\Rightarrow f'(x) > 2e^{2x} \Rightarrow f'(x) > 0 \ \forall x \in (0, \infty)$

49. If $g(x)=\int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\,dt$, then

[A] $g'\left(-\frac{\pi}{2}\right)=-2\pi$

[B] $g'\left(-\frac{\pi}{2}\right)=2\pi$

[C] $g'\left(\frac{\pi}{2}\right)=2\pi$

[D] $g'\left(\frac{\pi}{2}\right)=-2\pi$

Ans. (None is correct)

50. Let α and β be nonzero real numbers such that $2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1$. Then which of the following is/are true?

[A] $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

[B] $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

[C] $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

[D] $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

Ans. (A,C)

Sol. $2(\cos\beta-\cos\alpha)+\cos\alpha\cos\beta=1$. Let $\tan\frac{\beta}{2}=y,\ \tan\frac{\alpha}{2}=x$

$$2\left(\frac{1-y^2}{1+y^2}-\frac{1-x^2}{1+x^2}\right)+\left(\frac{1-x^2}{1+x^2}\right)\left(\frac{1-y^2}{1+y^2}\right)=1$$

$$\Rightarrow 2\left(\left(1-y^2\right)\left(1+x^2\right)-\left(1-x^2\right)\left(1+y^2\right)\right)+\left(1-x^2\right)\left(1-y^2\right)=\left(1+x^2\right)\left(1+y^2\right)$$

$$2\left(x^2-y^2\right)=x^2+y^2 \quad \Rightarrow x^2=3y^2 \Rightarrow x=\pm\sqrt{3}\,y$$

$$\tan\frac{\alpha}{2}=\pm\sqrt{3}\tan\frac{\beta}{2}$$

$$\tan\frac{\alpha}{2}-\sqrt{3}\tan\frac{\beta}{2}=0 \text{ and } \tan\frac{\alpha}{2}+\sqrt{3}\tan\frac{\beta}{2}=0$$

SECTION – 3 (Maximum Marks : 12)

- This section contains **TWO** paragraphs
- Based on each paragraph, there will be **TWO** questions
- Each question has **FOUR** option (A), (B), (C) and (D). **ONLY ONE** of these four option is correct.
- For each question, darken the bubble corresponding to all the correct option in the ORS.
- For each question, marks will be awarded in one of the following categories :

Full Marks : +3 If only the bubble corresponding to the correct option is darkened.

Zero Marks : 0 In all other cases.

Paragraph-1

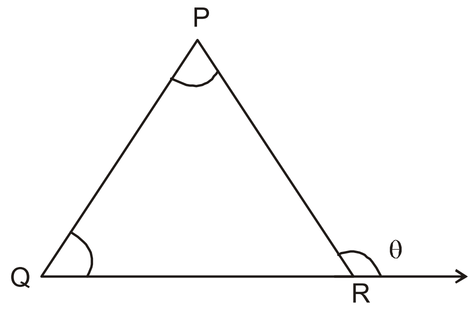
Let O be the origin, and $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ be three unit vectors in the directions of the sides $\overrightarrow{QR}, \overrightarrow{RP}, \overrightarrow{PQ}$, respectively, of a triangle PQR.

51. $\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| =$

[A] $\sin(P+R)$ [B] $\sin 2R$ [C] $\sin(P+Q)$ [D] $\sin(Q+R)$

Ans. **(C)**

Sol. In ΔPQR



Since $\angle P + \angle Q = \theta$

Now,

$\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| \Rightarrow |1||1| \sin(\angle P + \angle Q)|\hat{n}|$

$\left|\overrightarrow{OX} \times \overrightarrow{OY}\right| = \sin(P+Q)$

52. If the triangle PQR varies, then the minimum value of $\cos(P+Q)+\cos(Q+R)+\cos(R+P)$ is

[A] $-\frac{3}{2}$ [B] $\frac{3}{2}$ [C] $\frac{5}{3}$ [D] $-\frac{5}{3}$

Ans. **(A)**

Sol. Since, $\angle P + \angle Q + \angle R = \pi$

$\therefore \cos(P+Q)+\cos(Q+R)+\cos(R+P)$

$\Rightarrow \cos(\pi - R)+\cos(\pi - P)+\cos(\pi - Q)$

$\Rightarrow -(\cos P + \cos Q + \cos R)$

$\Rightarrow -\left[1 + 4\sin\frac{P}{2}\cdot\sin\frac{Q}{2}\cdot\sin\frac{R}{2}\right]$

Since minimum value will be at

$P = Q = R = \frac{\pi}{3}$

$$\therefore -\left[1+4\sin\frac{P}{2}\cdot\sin\frac{Q}{2}\cdot\sin\frac{R}{2}\right]\geq -\left[1+4\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right]\geq -\frac{3}{2}$$

$\therefore$ Minimum value will be $-\frac{3}{2}$

Paragraph-2

Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For n = 0, 1, 2,..., let $a_n = p\alpha^n + q\beta^n$

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then a = 0 = b.

53. If $a_4 = 28$, then p + 2q =

[A] 12 [B] 21 [C] 14 [D] 7

Ans. (A)

Sol. Given $x^2 - x - 1 = 0$

$$\alpha = \frac{1-\sqrt{5}}{2},\ \beta = \frac{1+\sqrt{5}}{2}$$

Also, $x^2 = x + 1$

$\therefore \alpha^2 = \alpha + 1,\ \beta^2 = \beta + 1$

$a_4 = p\alpha^4 + q\beta^4$ $\quad\Rightarrow 28 = p\left(\frac{1-\sqrt{5}}{2}\right)^4 + q\left(\frac{1+\sqrt{5}}{2}\right)^4$

$\Rightarrow 28\times 16 = (p+q) + (p-q)4\sqrt{5} + (p+q)30 + 20\sqrt{5}(p-q) + 25(p+q)$

$\Rightarrow 28\times 16 = 56(p+q) + 24\sqrt{5}(p-q)$

$\Rightarrow 56 = 7(p+q) + 3\sqrt{5}(p+q) \Rightarrow 7(p+q-8) + 3\sqrt{5}(p-q) = 0$

$\Rightarrow p+q-8 = p-q = 0$ $\quad\Rightarrow p = q = 4$ $\quad\therefore\ p + 2q = 12$

54. $a_{12} =$

[A] $a_{11} + 2a_{10}$ [B] $a_{11} + a_{10}$ [C] $a_{11} - a_{10}$ [D] $2a_{11} + a_{10}$

Ans. (B)

Sol. $A_{12} = p\alpha^{12} + q\beta^{12}$

$= p\alpha^{10}(\alpha+1) + q\beta^{10}(\beta+1)$ $\quad\left[\because x^2 = x+1\right]$

$= p\alpha^{11} + p\alpha^{10} + q\beta^{11} + q\beta^{10} = A_{11} + A_{10}$