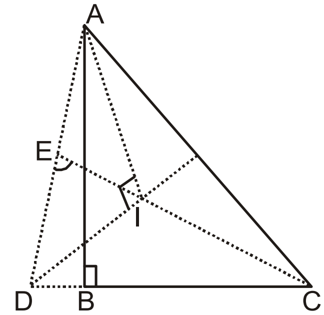
REGIONAL MATHEMATICAL OLYMPIAD - 2016
Paper with Solution

1. Let ABC be a right-angled triangle with $\angle B = 90^\circ$. Let I be the incentre of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and prove that $ID = \sqrt{b(b-a)}$, where BC = a and CA = b.

Sol. Join CI and produce it to meet AD at E. Since I is incentre



$\therefore \quad \angle AIC = 90^\circ + \frac{1}{2}\angle ABC = 135^\circ$

$\therefore \quad \angle AIE = 180^\circ - \angle AIC = 45^\circ = \angle EID$

and $\quad \angle CID = 180^\circ - \angle EID = 135^\circ$

Now $\Delta AIC \cong \Delta DIC$, since IC is common side,

$\angle ACI = \angle DCI$ and $\angle AIC = \angle CID$.

$\therefore \quad$ AC = DC and AI = DI

Thus CE is angle bisector in isosceles ΔACD and so $CE \perp AD$. Proved.

Now $AD^2 = AB^2 + BD^2$

$= (AC^2 - BC^2) + (CD - BC)^2$

$= (b^2 - a^2) + (b-a)^2 \qquad (\because DC = AC = b)$

$= 2b^2 - 2ab$

$\therefore \quad$ From right isosecels ΔAID,

$$ID = \frac{AD}{\sqrt{2}} = \sqrt{b^2 - ab} = \sqrt{b(b-a)}. \text{ Proved.}$$

2. Let a, b, c be positive real numbers such that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

Prove that $abc \le 1/8$.

Sol. Let $\alpha=\frac{a}{1+a}, \beta=\frac{b}{1+b}, \gamma=\frac{c}{1+c}$, then $\alpha+\beta+\gamma=1$ ... (i)

Now $\alpha=\frac{a}{1+a} \Rightarrow \alpha+a\alpha=a$

$\Rightarrow$ $a=\frac{\alpha}{1-\alpha}=\frac{\alpha}{\beta+\gamma}$ (using (i))

Similarly, $b=\frac{\beta}{\gamma+\alpha}$ and $c=\frac{\gamma}{\alpha+\beta}$

Now we have to prove that $abc \le \frac{1}{8}$ i.e. $\frac{\alpha}{\beta+\gamma}.\frac{\beta}{\gamma+\alpha}.\frac{\gamma}{\alpha+\beta} \le \frac{1}{8}$

i.e. $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta) \ge 8\alpha\beta\gamma \quad (\because \alpha,\beta,\gamma>0)$

Using A.M. – G.M. inequality,

$\frac{\alpha+\beta}{2} \ge \sqrt{\alpha\beta}$, $\frac{\beta+\gamma}{2} \ge \sqrt{\beta\gamma}$ and $\frac{\gamma+\alpha}{2} \ge \sqrt{\gamma\alpha}$

Multiplying the above three, we get

$$\frac{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}{8} \ge \sqrt{\alpha^2\beta^2\gamma^2}$$

i.e. $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha) \ge 8\alpha\beta\gamma$. Q.E.D.

3. For any natural number n, expressed in base 10, let S(n) denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

Sol. Let n has k digits, then

$$10^{k-1} \le n = 2(S(n))^2 \le 2(9k)^2 < 200k^2$$

$\Rightarrow$ $10^{k-3} < 2k^2$

Clearly $k \le 4$

$\therefore$ $S(n) \le 9k \le 36$

$\Rightarrow$ $n = 2(S(n))^2 \le 2592$

$\Rightarrow$ $S(n) \le 28$ ... (i)

(for n = 1999, S(n) = 28)

$\because$ both n and S(n) give the same remainder when divided by 9,

$\therefore$ we may assume $n = 9p + r$ and $S(n) = 9q + r$, where $p, q, r \in I$ and $r \in \{0, 1, ..., 8\}$.

Now, $n = 2(S(n))^2$

$\Rightarrow$ $9p + r = 2(9q + r)^2 = 162q^2 + 36qr + r^2$

$\Rightarrow$ $9 | r - r^2$

$\Rightarrow$ $r = 0$ or 5 ... (ii)

$\therefore$ possible values of S(n) are 5,9,14,18,23 and 27 and corresponding values of n are 50, 162, 392, 648, 1058 and 1458. Out of which only 50, 162, 392 and 648 satisfy the condition $n = 2(S(n))^2$ and so are the required numbers.

4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

Sol. Case I : Let the first place be filled with an odd digit, then out of the remaining 5 places two places can be choosen in ${}^5C_2$ ways. Now these three places can be filled with odd digits in 5 × 5 × 5 i.e. 5^3 ways and remaining three places can be filled with even digits in 5 × 5 × 5 i.e 5^3 ways. So total no. of ways in this case = ${}^5C_2 \times 5^3 \times 5^3 = 10 \times 5^6$.

CASE II : Let the first place is filled with an even digit, then out of remaining five places two places can be chosen in ${}^5C_2$ ways. The three places can be filled with even digits in 4 × 5 × 5 ways and remaining three with odd digits in 5 × 5 × 5 ways.

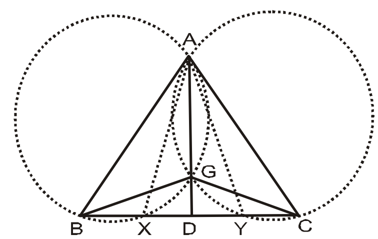
So total number of ways in this case = ${}^5C_2 \times 4 \times 5^2 \times 5^3 = 8 \times 5^6$

Hence total number of required numbers

$= 10 \times 5^6 + 8 \times 5^6 = 18 \times 5^6 = 281250$

5. Let ABC be a triangle with centroid G. Let the circumcircle of triangle AGB intersect the line BC in X different from B; and the circumcircle of triangle AGC intersect the line BC in Y different from C. Prove that G is the centroid of triangle AXY.

Sol. Let AG produced meet BC at D



$\because$ G is centroid of $\triangle ABC$

$\therefore$ AG : GD = 2 : 1 ... (i)

We have to prove that G is also centorid of $\triangle AXY$.

For this, due to (i), it is enough to prove that D is mid point of XY.

$\because$ DB and DA are secants of the circumcircle of $\triangle AGB$

$\therefore$ $DX \cdot DB = DG \cdot DA$ ... (ii)

Similarly using circumcircle of $\triangle AGC$,

$DY.DC = DG.DA$ ... (iii)

From (ii) & (iii), $DX \cdot DB = DY \cdot DC$

But $DB = DC$ ($\because AD$ is median of $\triangle ABC$)

Hence DX = DY. Q.E.D.

6. Let $(a_1, a_2, a_3, ...)$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in geometric progression.

Sol. Let d be the common difference of the A.P., then $d \in I^+$.

Let x and x + md ($m \in I^+$) be any two terms of the A.P. such that $\frac{x + md}{x} = \lambda$.

Now we choose m so that λ becomes independent of x.

Taking m = x, we get $\lambda = \frac{x + xd}{x} = 1 + d$

Thus for any given term x of the A.P., x + xd i.e. x (1 +d) is also a term of the A.P.

For each $k \in N$, let $b_k = a_1(1 + d)^{k-1}$, then

Clearly $b_1, b_2, b_3, ...$ are in G.P. such that

$$\begin{aligned} b_k &= a_1(1 + (k-1)d + ... + d^{k-1}) \quad \text{(Using binomial expansion)} \\ &= a_1 + (m-1)d \text{ for some } m \in I^+ \\ &= a_m \end{aligned}$$

Thus b_1, b_2, b_3 ... is an infinite subsequence of $a_1, a_2, a_3, ...$ which is in geometric progression. Proved.