32nd INDIAN NATIONAL MATHEMATICAL OLYMPIAD (INMO)-2017

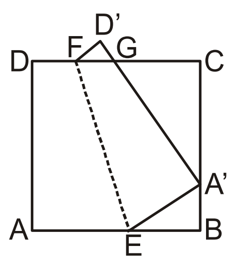
SOLUTIONS

Time: 4 hours **January 15, 2017**

Instructions :

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- Answer all the questions.
- Answer to each question should start on a new page. Clearly indicate the question number.

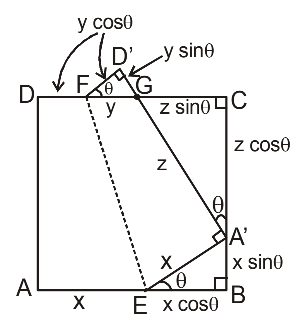
1. In the given figure, ABCD is a square sheet of paper. It is folded along EF such that A goes to a point A' different from B and C, on the side BC and D goes to D'. The line A'D' cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles GD'F and A'BE.



Sol.: Let A'E = AE = x, FG = y, A'G = z

and $\angle A'EB = \theta$,

Clearly $\angle A'EB = \angle CA'G = \angle D'FG = \theta$



$\therefore \quad \Delta A'BE \sim \Delta GCA' \sim \Delta D'FG$

$\because$ inradii of similar triangles are proportional to their corresponding sides.

$\therefore$ in order to prove that the sum of inradii of triangles GD'F and A'BE equals that of $\Delta GCA'$, it is enough to prove that x + y = z

Now,

AB = BC = CD = A'D'

$\Rightarrow$ AE + EB = BA' + A'C = DF + FG + GC = A'G + GD'

$\Rightarrow$ x + x cosθ = x sinθ + z cosθ = y cos θ + y + z sin θ = z + y sin θ

$\Rightarrow$ x(1 + cosθ – sinθ) = z cosθ and y (1 + cosθ – sinθ) = z (1 – sin θ)

adding the two relations,

(x + y) (1 + cosθ – sinθ) = z (1 + cosθ – sinθ)

but 0 < θ < 90º $\Rightarrow$ cosθ + (1 – sinθ) > 0

$\therefore$ x + y = z. Q.E.D.

2. Suppose $n \geq 0$ is an integer and all the roots of $x^3 + ax + 4 - \left(2 \times 2016^n\right) = 0$ are integers. Find all possible values of α.

Sol.: Let $p \leq q \leq r\left(p, q, r \in I\right)$ be the roots of the equation $x^3 + \alpha x + 4 - 2 \times 2016^n = 0,$ then $p + q + r = 0,$ $pq + qr + rp = \alpha$ and $pqr = 2 \times 2016^n - 4$

If n = 0, then pqr = – 2 and p + q + r = 0

which gives (p, q, r) = (–2, 1, 1) and so α = – 3.

Now, we assume $n \geq 1,$ then

$$pqr = 2 \times 2016^n - 4 > 0$$

but $p \leq q \leq r$ and $p + q + r = 0$

$\therefore$ p < 0, q < 0 and r > 0

Let p = –a, q = –b and r = a + b, where $a, b \in I^+,$

then $2 \times 2016^n - 4 = ab\left(a + b\right)$

$\because$ 7|2016

$\therefore$ $2 \times 2016^n - 4 \equiv -4 \bmod(7) \equiv 3 \bmod(7)$

$\Rightarrow$ $ab\left(a + b\right) \equiv 3 \bmod(7)$

Let $a = 7k_1 + r_1$ and $b = 7k_2 + r_2,$ where $k_1, k_2 \in W$ and $r_1, r_2 \in \{0, 1, 2, ..., 6\}.$

Now $ab\left(a + b\right) \equiv 3 \bmod(7)$

$\Rightarrow$ $r_1 r_2\left(r_1 + r_2\right) \equiv 3 \bmod(7)$ ....(1)

Clearly $r_1 \neq 0, r_2 \neq 0$ and $r_1 + r_2 \neq 0$ or $7.$

If $r_1 = r_2,$ then $2r_1^3 \equiv 3 \bmod(7)$

but $r_1^3 \equiv \pm 1 \bmod(7)$ $\forall r_1 \in \{1, 2, ..., 6\}$

$\therefore$ $2r_1^3 \equiv \pm 2 \bmod(7)$ $\forall r_1 \in \{1, 2, ..., 6\}$

Hence $r_1 \neq r_2$

From symmetry of (1), it is enough to consider $r_1 < r_2$.

r_1	r_2	$r_1 r_2(r_1+r_2)$	mod (7)
1	2	6	6
1	3	12	5
1	4	20	6
1	5	30	2
2	3	30	2
2	4	48	6
2	6	96	5
3	5	120	1
3	6	162	1
4	5	180	5
4	6	240	2
5	6	330	1

Thus equation (1) can not hold for any possibe combination of r_1 and r_2 and hence the only possible value of α is – 3.

3. Find the number of triples (x, a, b), where x is a real number and a, b belong to the set {1, 2, 3, 4, 5, 6, 7, 8, 9} such that

$$x^2 - a\{x\} + b = 0$$

where {x} denotes the fractional part of the real number x. (For example {1. 1} = 0.1 = {–0.9}.)

<u>Sol</u>.: Let $x = n + \lambda$, where $n \in I$ and $0 \le \lambda < 1$, then $\{x\} = \lambda$.

Now, $x^2 - a\{x\} + b = 0$ .....(1)

$\Leftrightarrow (n+\lambda)^2 - a\lambda + b = 0$

$\Leftrightarrow \lambda^2 - (a-2n)\lambda + (n^2+b) = 0$ ......(2)

Clearly $\lambda \ne 0$

Let $f(t) = t^2 - (a-2n)t + n^2 + b,\ t \in R$

Let $f(t) = 0$ has a root $\lambda \in (0, 1)$.

But product of zeros of f(x) $= n^2 + b \ge 1$

$\therefore$ The other zero of f(x) must be greater than 1 and so f (x) will have exactly one zero in (0, 1)

$\Leftrightarrow f(0)f(1) < 0$

$\Leftrightarrow (n^2+b)(1+(a-2n)+n^2+b) < 0$

$\Leftrightarrow (n+1)^2 < a - b$ .....(3)

Conversely for any triplet (n, a, b) satisfying (3), we get a unique $\lambda \in (0,1)$ satisfying (2) and hence a unique triplet (x, a, b) satisfying (1).

Thus number of triplets (x, a, b) satisfying (1) is same as that of (n, a, b) satisfying (3) which we find as follows.

$$(n+1)^2 < a - b \le 8 \ \ (\because 1 \le a, b \le 9)$$

$\Rightarrow \ -2\sqrt{2} \le n + 1 \le 2\sqrt{2}$

$\Rightarrow \ -3 \le n \le 1$

For n = – 3 or 1, a – b > 4 i.e. a > b + 4, which has 4 + 3 + 2 + 1 i.e. 10 solutions.

For n = – 2 or 0, a > b + 1, which has 7 + 6 + ... + 1 i.e. 28 solutions.

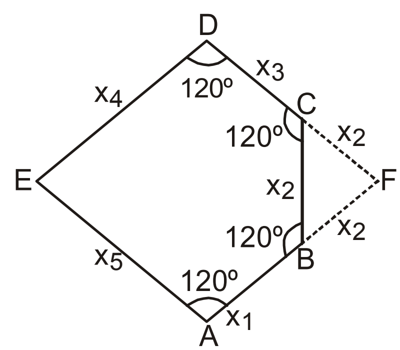
For n = – 1, a > b, which has 8 + 6 + ... + 1 i.e. 36 solutions.

Thus total no. of solutions = 2 × 10 + 2 × 28 + 36 = 112.

4. Let ABCDE be a convex pentagon in which $\angle A = \angle B = \angle C = \angle D = 120^\circ$ and side lengths are five consecutive integers in some order. Find all possible values of AB + BC + CD.

Sol.: Let AB = x_1, BC = x_2, CD = x_3, DE = x_4 and EA = x_5.

Produce AB and DC to meet at F, then



$$\angle CBF = 180^\circ - \angle B = 180^\circ - 120^\circ = 60^\circ$$

and $\angle FCB = 180^\circ - \angle C = 60^\circ$

$\therefore$ ΔBFC is equilateral.

Hence $\angle F = 60^\circ$ and $BF = FC = BC = x_2$

$\because$ $\angle D + \angle F = 180^\circ$ and $\angle A + \angle F = 180^\circ$

$\therefore$ $AF \,||\, DE$ and $AE \,||\, DF$

$\therefore$ AFDE is a parallelogram.

$\therefore$ AF = ED and DF = AE

$\Rightarrow$ $x_1 + x_2 = x_4$ and $x_2 + x_3 = x_5$ .....(1)

But $x_1, x_2, ..., x_5$ are five consecutive integers in some order.

$\therefore$ $x_1 + x_2 = x_4 \le x_1 + 4$

$\Rightarrow$ $x_2 \le 4$

Similarly $x_1 \le 4$ and $x_3 \le 4$ and So, $x_4, x_5 \le 6$.

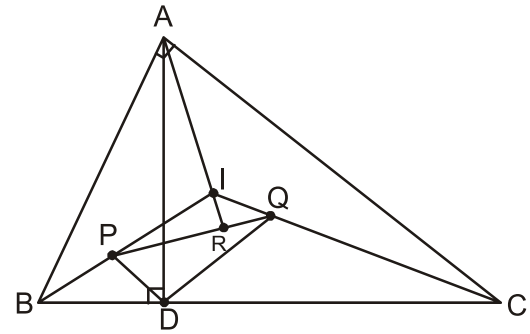
Thus $\{x_1, x_2, ..., x_5\} = \{1, 2, ..., 5\}$ or $\{2, 3, ..., 6\}$

Hence, due to (1), the possible solutions are given by

$$(x_1, x_2, ..., x_5) = (2,1,4,3,5),\ (4,1,2,5,3), (1,3,2,4,5), (2,3,1,5,4), (3,2,4,5,6) \text{ or } (4,2,3,6,5)$$

$\therefore$ AB + BC + CD = $x_1 + x_2 + x_3$ = 6, 7 or 9.

5. Let ABC be a triangle with $\angle A = 90^o$ and AB < AC. Let AD be the altitude from A on to BC. Let P, Q and I denote respectively the incentres of triangles ABD, ACD and ABC. Prove that AI is perpendicular to PQ and AI = PQ.



Sol.: $\because$ P, I lie on angle bisector of $\angle B$

$\therefore$ B, P, I are collinear

Similarly C, Q, I are collinear.

Join PD, DQ, BI and CI. Produce AI to meet PQ at R.

$\because$ PD and DQ are bisectors of angles $\angle ADB$ and $\angle ADC$

$\therefore$ $\angle PDQ = 90^o$

Again, $\because$ $\Delta ADB \sim \Delta CDA \sim \Delta CAB$

$\therefore$ the distances of the incentres of the three triangles from their corresponding vertices will be proportional to their corresponding sides.

$$\therefore \quad \frac{PD}{AB} = \frac{QD}{AC} = \frac{AI}{BC} = \frac{\sqrt{PD^2 + QD^2}}{\sqrt{AB^2 + BC^2}} = \frac{PQ}{BC}$$

$\therefore$ $\Delta PDQ \sim \Delta BAC$ and $AI = PQ$.

$\therefore \angle DPQ = \angle B$

Now, $\angle IPD = \angle PBD + \angle PDB = \frac{1}{2}\angle B + 45^o$

$$\therefore \quad \angle IPR = \angle IPD - \angle DPQ = \frac{1}{2}\angle B + 45^o - \angle B$$

Also, $\angle AIB = 90^o + \frac{1}{2}\angle C$

$$\therefore \quad \angle ARP = \angle AIB - \angle IPR = 90^o + \frac{1}{2}\angle C - \left(45^o - \frac{1}{2}\angle B\right) = 90^o + \frac{1}{2}(\angle B + \angle C) - 45^o = 90^o.$$

Q.E.D.

6. Let $n \geq 1$ be an integer and consider the sum

$$x = \sum_{k \geq 0} \binom{n}{2k} 2^{n-2k} 3^k = \binom{n}{0} 2^n + \binom{n}{2} 2^{n-2}.3 + \binom{n}{4} 2^{n-4}.3^2 + \ldots$$

Show that $2x - 1, 2x, 2x + 1$ form the sides of a triangle whose area and inradius are also integers.

Sol.: Here $x \in I$ and $x = {}^nC_0.2^n + {}^nC_2.2^{n-2}\sqrt{3}^2 + {}^nC_4.2^{n-4}.\sqrt{3}^4 + \ldots$

$$= \frac{1}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n\right] \geq 2 \ \forall n \geq 1$$

$\therefore$ $2x - 1 + 2x > 2x + 1$

$\therefore$ $2x-1, 2x, 2x+1$ form sides of a triangle

whose semiperimeter $s=\dfrac{2x-1+2x+2x+1}{2}=3x$

$\therefore$ $\text{Area}=\sqrt{3x\left(3x-(2x-1)\right)(3x-2x)\left(3x-(2x+1)\right)}$

$$=\sqrt{3x^2\left(x^2-1\right)}$$

$$=\frac{x\sqrt{3}}{2}\sqrt{\left((2x)^2-4\right)}$$

$$=\frac{x\sqrt{3}}{2}\sqrt{\left(\left(a+\frac{1}{a}\right)^2-4\right)},\ \text{where } a=\left(2+\sqrt{3}\right)^n$$

$$=\frac{x\sqrt{3}}{2}\sqrt{\left(a-\frac{1}{a}\right)^2}$$

$$=\frac{x\sqrt{3}}{2}\left(a-\frac{1}{a}\right)=\frac{x\sqrt{3}}{2}\left[\left(2+\sqrt{3}\right)^n-\left(2-\sqrt{3}\right)^n\right]$$

$$=x\sqrt{3}\left[{}^nC_1 2^{n-1}\cdot\sqrt{3}+{}^nC_3\cdot 2^{n-3}\cdot\sqrt{3}^3+{}^nC_5\cdot 2^{n-5}\cdot\sqrt{3}^5+....\right]$$

$$=3x\left[{}^nC_1\,2^{n-1}+{}^nC_3\,2^{n-3}\cdot 3+{}^nC_5\,2^{n-5}\cdot 3^2+....\right]$$

Which is clearly an integer.

Also inradius $=\dfrac{\Delta}{s}=\dfrac{\Delta}{3x}={}^nC_1\cdot 2^{n-1}+{}^nC_3\cdot 2^{n-3}\cdot 3+{}^nC_5\cdot 2^{n-5}\cdot 3^2+....$

which is also an integer. Q.E.D.