PRMO 2017-18 : QUESTIONS & SOLUTIONS

1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3 ?

Soln. (28)

Let n be the positive integer less than 1000 and s be the sum of its digits, then

3|n and 7|s.

$3\,|\,n \;\Rightarrow\; 3\,|\,s$

$\therefore\; 21\,|\,s$

Also n < 1000 $\Rightarrow s \le 27$

$\therefore s = 21$

Clearly, n must be a 3 digit number.

Let x_1, x_2, x_3 be the digits, then

$x_1 + x_2 + x_3 = 21$ …(1)

where $1 \le x_1 \le 9, 0 \le x_2, x_3 \le 9$

$\Rightarrow\; x_2 + x_3 = 21 - x_1 \le 18$

$\Rightarrow\; x_1 \ge 3$

For x_1 = 3, 4, ..., 9, the equation (1) has 1, 2, 3, ..., 7 solutions

$\therefore$ total possible solution of equation (1)

$$= 1 + 2 + \ldots + 7 = \frac{7 \times 8}{2} = 28$$

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183, a\sqrt{b} + b\sqrt{a} = 182$. Find $\frac{9}{5}(a+b)$.

Soln. (73)

$a\sqrt{a} + b\sqrt{b} = 183$ …(1)

$a\sqrt{b} + b\sqrt{a} = 182$ …(2)

From (1) & (2),

$$\left(\sqrt{a}\right)^3 + \left(\sqrt{b}\right)^3 + 3\sqrt{a}\sqrt{b}\left(\sqrt{a} + \sqrt{b}\right) = 183 + 3 \times 182$$

$$\Rightarrow \left(\sqrt{a} + \sqrt{b}\right)^3 = 729$$

$\Rightarrow \sqrt{a} + \sqrt{b} = 9$ …(3)

$\therefore$ From (2),

$$\sqrt{a}\sqrt{b}\left(\sqrt{a} + \sqrt{b}\right) = 182$$

$\sqrt{a}\sqrt{b} = \frac{182}{9}$ …(4)

From (3) & (4),

$$\frac{9}{5}(a+b) = \frac{9}{5}\left\{\left(\sqrt{a} + \sqrt{b}\right)^2 - 2\sqrt{a}\sqrt{b}\right\}$$

$$= \frac{9}{5}\left\{81 - 2 \times \frac{182}{9}\right\} = 73$$

3. A contractor has two teams of workers : team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job ?

Soln. (16)

Team A's one day work $=\frac{1}{12}$

Team B's one day work $=\frac{1}{36}$

Work done by team A in 6 days $=\frac{1}{12}\times 6=\frac{1}{2}$

Work done by team B in 2 days $=\frac{1}{36}\times 2=\frac{1}{18}$

Remaining work after team A left $=1-\left(\frac{1}{2}+\frac{1}{18}\right)=\frac{4}{9}$

$\therefore$ No. of days required for team B to complete the remaining work $=\frac{4/9}{1/36}=16$

4. Let a, b be integers such that all the roots of the equation $(x^2+ax+20)(x^2+17x+b)=0$ are negative integers. What is the smallest possible value of a + b ?

Soln. (25)

Let $\alpha \le \beta < 0$ be the roots of $x^2 + ax + 20 = 0$ and $\gamma \le \delta < 0$ be the roots of $x^2 + 17x + b = 0$,

then $\alpha\beta = 20, \alpha+\beta = -a$

$\gamma\delta = b,\ \gamma+\delta = -17$

$\therefore$ possible values of (α,β) are (– 20, –1), (–10, –2), (–5, –4)

and that of (γ,δ) are (–16, –1), (–15, –2),...,(–9, –8).

$\therefore$ Smallest value of a = –(–5–4) = 9 and that of b = (–16) (–1) = 16

$\therefore$ smallest value of a + b = 9 +16 = 25.

5. Let u, v, w be real numbers in geometric progression such that u > v > w. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

Soln. (48)

u, v, w are in G.P.

$\Rightarrow v^2 = uw \Rightarrow |v| = \sqrt{uw}$

$u^{40} = v^n = w^{60}$

$u^{40} = w^{60} \Rightarrow |w| = u^{40/60} = u^{2/3}$

$\therefore u^{40} = v^n = \left(\sqrt{uw}\right)^n$

$= \left(u \cdot u^{2/3}\right)^{n/2} = u^{5n/6}$

$\therefore \frac{5n}{6} = 40 \Rightarrow n = 48$

6. Let the sum $\sum_{n=1}^{9}\frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of q – p.

Soln. (83)

$$\sum_{n=1}^{9}\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\sum_{n=1}^{9}\frac{(n+2)-n}{n(n+1)(n+2)}$$

$$=\frac{1}{2}\sum_{n=1}^{9}\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)$$

$$=\frac{1}{2}\left[\left(\frac{1}{1.2}-\frac{1}{2.3}\right)+\left(\frac{1}{2.3}-\frac{1}{3.4}\right)+...+\left(\frac{1}{9.10}-\frac{1}{10.11}\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{110}\right)=\frac{27}{110}=\frac{p}{q}$$

$\therefore q-p=110-27=83$

7. Find the number of positive integers n, such that $\sqrt{n}+\sqrt{n+1}<11$.

Soln. (29)

$\sqrt{n}+\sqrt{n+1}<11$ …(1)

$\Rightarrow \sqrt{n+1}<11-\sqrt{n}$

squaring both sides,

$n+1<121+n-22\sqrt{n}$

$\Rightarrow 22\sqrt{n}<120$

$\Rightarrow \sqrt{n}<\frac{60}{11}\Rightarrow n<\frac{3600}{121}$

$\therefore\ n \le 29$

$\therefore$ number of positive integers satisfying (1) is 29.

8. A pen costs ₹ 11 and a notebook costs ₹ 13. Find the number of ways in which a person can spend exactly ₹ 1000 to buy pens and notebooks.

Soln. (7)

Let the person buys x pens and y notebooks, then

11x + 13y = 1000 …(1)

$\Rightarrow 11x=1000-13y=(1001-11y)-(2y+1)$ = 11 (91 – y) – (2y + 1)

$\Rightarrow 11|2y+1$

Let 2y + 1 = 11 (2k – 1), $k\in I^{+}$

$\Rightarrow y=11k-6$

$\therefore 11x=11(97-11k)-11(2k-1)$

$\Rightarrow x=98-13k$

But x > 0 $\Rightarrow k<\frac{98}{13}\Rightarrow k\le 7$

$\therefore$ for each $k\in\{1,2,....,7\}$, we get a unique pair (x, y) = (98 – 13k, 11k –6) satisfying equation (1)

Hence 7 ways are possible.

9. There are five cities A, B,C, D,E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once ? (The order in which he visits the cities also matters : e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)

Soln. (60)

Each possible route gives a permutation of (B, C, D, E) taken two or more at a time and vice-versa.

$\therefore$ required no. of ways

$= {}^4P_4 + {}^4P_3 + {}^4P_2$

$= \lfloor 4 + \frac{\lfloor 4}{\lfloor 1} + \frac{\lfloor 4}{\lfloor 2} = 24 + 24 + 12 = 60$

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated ?

Soln. (48)

The rooms can be selected in only two possible ways as follows :

G		G	
	G		G

or

	G		G
G		G	

In each case guests can be accommodated in $\lfloor 4$ ways.

$\therefore$ required number of ways $= 2\lfloor 4 = 48$.

11. Let $f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$ for all real x. Find the least natural number n such that $f(n\pi + x) = f(x)$ for all real x.

Soln. (60)

$f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$

period of $\sin\frac{x}{3} = 2\pi \times 3 = 6\pi$

period of $\cos\frac{3x}{10} = \frac{10}{3} \times 2\pi = \frac{20\pi}{3}$

$\therefore$ period of f(x) = L.C.M. of $\left(6\pi, \frac{20\pi}{3}\right)$

$\therefore$ least value of n satisfying $f(n\pi + x) = f(x)$ is 60.

12. In a class, the total numbers of boys and girls are in the ratio 4 : 3. On one day it was found that 8 boys and 14 girls were absent from the class, and that the number of boys was the square of the number of girls. What is the total number of students in the class ?

Soln. (42)

Let the number of boys and girls in the class be 4x and 3x respectively, then according to question,

$4x - 8 = (3x - 14)^2$

$\Rightarrow 4x - 8 = 9x^2 - 84x + 196$

$\Rightarrow 9x^2 - 88x + 204 = 0$

$\Rightarrow (9x - 34)(x - 6) = 0$

$\Rightarrow x = \frac{34}{9}$ or 6

But $x \in N$, $\therefore x = 6$

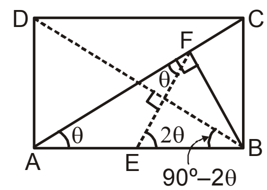
$\therefore$ total no. of students in class = 4x + 3x = 42.

13. In a rectangle ABCD, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find AB.

Soln. (24)

Let $\angle BAC = \theta$

Since E is mid point of hypotenous AB of right ΔAFB, therefore
AE = FE = BE



$\therefore \angle EFA = \angle FAE = \theta$

and $\angle FEB = \angle EAF + \angle EFA = 2\theta$

$\Rightarrow \angle EBD = 90° - \angle BEF = 90° - 2\theta$

But $\angle FAE = \angle CAB = \angle DBA$

$\therefore \theta = 90° - 2\theta \Rightarrow \theta = 30°$

$\therefore$ in ΔABC, $\tan\theta = \dfrac{BC}{AB}$

$\Rightarrow AB = BC\cot\theta = 8\sqrt{3}\cot 30°$ = 24.

14. Suppose x is a positive real number such that {x}, [x] and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here [x] denotes the integer part of x and {x} = x – [x].)

Soln. (10)

A/Q, $[x]^2 = \{x\}x$, x > 0
Let [x] = m and {x} = f, where $m \in W$ and $0 \le f < 1$,

then $m^2 = f \cdot (m+f)$ $(\because x = m+f)$

$\Rightarrow m^2 - mf - f^2 = 0$

$m = 0 \Rightarrow f = 0 \Rightarrow x = 0$

$\therefore m \ne 0$

$\therefore \left(\dfrac{f}{m}\right)^2 + \left(\dfrac{f}{m}\right) - 1 = 0$

$\Rightarrow \dfrac{f}{m} = -\dfrac{1 \pm \sqrt{5}}{2}$

But $\dfrac{f}{m} > 0$ $\quad \therefore \dfrac{f}{m} = \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow f = \dfrac{\sqrt{5}-1}{2}m$

But $0 \le f < 1$

$\therefore 0 \le \dfrac{\sqrt{5}-1}{2}m < 1$

$\Rightarrow 0 \le m < \dfrac{2}{\sqrt{5}-1} = \dfrac{\sqrt{5}+1}{2}$

$\Rightarrow m = 1$ $\qquad (\because m \ne 0)$

$\therefore f = \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow x = m + f = \frac{\sqrt{5}-1}{2} + 1 = \frac{\sqrt{5}+1}{2}$

Now $x^n > 100 \Rightarrow \left(\frac{\sqrt{5}+1}{2}\right)^n > 100$

$\because \frac{\sqrt{5}+1}{2} < \sqrt{3}$ and $\left(\sqrt{3}\right)^8 = 81$ $\therefore n > 8$

Now $\left(\frac{\sqrt{5}+1}{2}\right)^9 < 100 < \left(\frac{\sqrt{5}+1}{2}\right)^{10}$

$\therefore$ least value of n is 10.

15. Integers 1, 2, 3......n, where n > 2, are written on a board. Two numbers m, k such that 1 < m < n, 1 < k < n are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers ?

Soln. (51)

A/Q, $\frac{(1+2+3+...+n)-(m+k)}{n-2} = 17$

$\Rightarrow \frac{n(n+1)}{2} - (m+k) = 17(n-2)$

$\Rightarrow m + k = \frac{n(n+1)}{2} - 17(n-2) = \frac{n^2 - 33n + 68}{2}$

But $5 \le m + k \le 2n - 3$

$\therefore 5 \le \frac{n^2 - 33n + 68}{2} \le 2n - 3$

$\Rightarrow n^2 - 33n + 58 \ge 0$ and $n^2 - 37n + 74 \le 0$

$\Rightarrow (n \le 1 \text{ or } n \ge 32) \text{ and } (3 \le n \le 34)$

$\Rightarrow 32 \le n \le 34$

$n = 32 \Rightarrow m + k = 18$

$n = 33 \Rightarrow m + k = 34$

and $n = 34 \Rightarrow m + k = 51$

$\therefore$ maximum value of m + k is 51.

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

Soln. (36)

Let the numbers be a, ar, ar^2, ar^3, ar^4, where $a \in N, 10 \le a < 99$ and $r \in Q$, $r > 1$.

$\because ar^4 < 100$ $\therefore r^4 < \frac{100}{a} \le 10$ $\Rightarrow r < 2$

Let $r = \frac{p}{q}$ in lowest form

then $ar^4 \in I$ $\Rightarrow q^4 | a$

$\therefore q = 2$ or 3 $(\because 1 < q^4 < a < 100)$

For q = 2, $a = 16k$, $k \in I^+$

$\therefore\ ar^4 = k.p^4 < 100$

$\therefore\ p = 3 \qquad (\because r > 1,\ \therefore p > q)$

and $k \cdot 3^4 < 100 \ \Rightarrow k = 1$

For q = 3, $a = 81m,\ m \in I^+$

$\therefore\ ar^4 = m \cdot p^4 < 100$

$\Rightarrow p \le 3$

but p > q = 3

$\therefore$ no value of p in possible.

Hence only possible solution is a = 16 and $r = \frac{3}{2}$

$\therefore$ Middle term $= ar^2 = 16 \times \frac{9}{4} = 36$

17. Suppose the altitudes of a triangle are 10, 12 and 15. What is its semi-perimeter ?

Soln. (Bonus)

Here h_a = 10, h_b = 12, h_c = 15

$$\Delta = \frac{1}{2} \times a \times h_a = \frac{1}{2} \times b \times h_b = \frac{1}{2} \times c \times h_c$$

$$\Rightarrow\ 2\Delta = 10a = 12b = 15c = 60\lambda \ \text{(say)}$$

$$\Rightarrow\ \Delta = 30\lambda, a = 6\lambda, b = 5\lambda \ \text{and}\ c = 4\lambda$$

$$\therefore\ s = \frac{a+b+c}{2} = \frac{15\lambda}{2}$$

$$\therefore\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow\ 30\lambda = \sqrt{\frac{15}{2}\lambda \times \frac{3}{2}\lambda \times \frac{5}{2}\lambda \times \frac{7}{2}\lambda}$$

$$\Rightarrow\ \lambda = \frac{30 \times 4}{15 \times \sqrt{7}} = \frac{8}{\sqrt{7}}$$

$$\therefore\ s = \frac{15}{2} \cdot \frac{8}{\sqrt{7}} = \frac{60}{\sqrt{7}}$$

18. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and xy + 4yz + 2zx = 24 what is the value of $x^2 + y^2 + z^2$?

Soln. (21)

$x^2 + 4y^2 + 16z^2 = 48$ …(1)

$xy + 4yz + 2zx = 24$ …(2)

From (1) & (2),

$x^2 + 4y^2 + 16z^2 - 2(xy + 4yz + 2zx) = 0$

$$\Rightarrow\ \frac{1}{2}\{(x-2y)^2 + (2y-4z)^2 + (4z-x)^2\} = 0$$

$$\Rightarrow\ x = 2y = 4z$$

Now from (1),

$$3x^2 = 48 \ \Rightarrow x^2 = 16$$

$$\therefore\ x^2 + y^2 + z^2 = x^2 + \frac{x^2}{4} + \frac{x^2}{16}$$

= 16 + 4 + 1 = 21

19. Suppose 1, 2, 3 are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.

Soln. (36)

Let 4th root be α, then

sum of roots $= 1 + 2 + 3 + \alpha = 0$

$\Rightarrow \alpha = -6$

$\therefore$ Product of roots $= 1 \cdot 2 \cdot 3 \cdot (-6) = -c$

$\Rightarrow c = 36$

20. What is the number of triples (a, b, c) of positive integers such that (i) a < b < c < 10 and (ii) a, b, c, 10 form the sides of a quadrilateral ?

Soln. (73)

$\because a < b < c < 10$

$\therefore$ a,b,c,10 form a quadrilateral iff a + b + c > 10

i.e. $a + b + c \geq 11$

Now $11 \leq a + b + c \leq c - 2 + c - 1 + c$

$\Rightarrow 3c - 3 \geq 11 \Rightarrow c \geq 5$

$\therefore 5 \leq c \leq 9$

For c = 5, $a + b \geq 6$ and $1 \leq a < b \leq 4$

which has only two solutions (2, 4) & (3, 4).

For c = 6, $a + b \geq 5$ and $1 \leq a < b \leq 5$

which has ${}^5C_2 - 2 = 8$ solutions.

For c = 7, $a + b \geq 4$ and $1 \leq a < b \leq 6$

which has ${}^6C_2 - 1 = 14$ solutions.

For c = 8, $a + b \geq 3$ and $1 \leq a < b \leq 7$

which has ${}^7C_2 = 21$ solutions.

For c = 9, $a + b \geq 2$ and $1 \leq a < b \leq 8$

which ${}^8C_2 = 28$ solutions.

$\therefore$ Total no. of solutions is 73.

21. Find the number of ordered triples (a, b, c) of positive integers such that abc = 108.

Soln. (60)

abc = 108 = $2^2 \times 3^3$

number of triplets (a, b, c)

= number of ways of distributing two 2's and three 3's among a, b, c

= ${}^{2+2}C_2 \times {}^{3+2}C_2$

= 6 × 10 = 60 .

22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed ?

Soln. (36)

Let there be $n(n \geq 2)$ lines in the plane. Now introducing a new line will cut each of the existing n lines and hence will form n – 1 new polygons at most.

$\therefore$ Maximum possible number of polygons formed with the 10 lines

$$= 1 + 2 + ... + 8 = \frac{8 \times 9}{2} = 36$$

23. Suppose an integer r, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p.

Soln. (47)

$7x^2 - 44x + 12 = p^n$

$\Rightarrow (7x-2)(x-6) = p^n$

Let m = x – 6, then (7m + 40) m = p^n …(1)

If $|m| > |7m+40|$, then

$(7m + 40)^2 - m^2 < 0 \Rightarrow (8m+40)(6m+40) < 0$

$\Rightarrow -\frac{20}{3} < m < -5 \quad \Rightarrow m = -6$

But m = –6 does not satisfy (1)

$|m| \le |7m+40|$

But both must be powers of p or 1.

$\therefore m \mid 7m+40$

$\Rightarrow m \mid 40$

$\because |m|$ is power of a prime or 1

$\therefore m = \pm 1, \pm 2, \pm 4, \pm 8, \pm 5$

Of these only m = 1 or –8 satisfy (1).

For m = 1, 47 = p^n $\Rightarrow p = 47, n = 1$

For m = – 8, 128 = p^n $\Rightarrow p = 2, n = 7$

$\therefore$ Largest value of p = 47.

24. Let P be an interior point of a triangle ABC whose side lengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

Soln. (Bonus)

Clearly PKBT, PMCL and PSAN are parallelograms.

Let PT = KB = x, PM = LC = y,

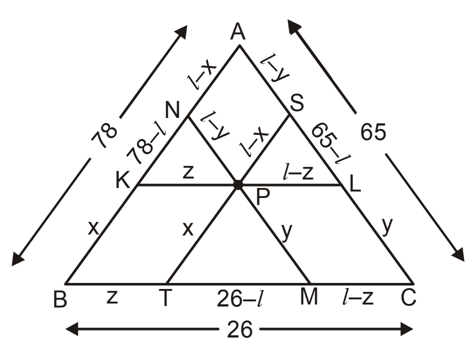
PK = BT = z and KL = MN = ST = ℓ

$\because \Delta PTM \sim \Delta ABC\phi$

$\therefore \frac{y}{65} = \frac{26-\ell}{26}$ …(1)

Again, $\Delta NKP \sim \Delta ABC$

$\Rightarrow \frac{\ell - y}{65} = \frac{78-\ell}{78}$ …(2)



Adding (1) & (2),

$\frac{\ell}{65} = \frac{26-\ell}{26} + \frac{78-\ell}{78}$

$= \frac{78 - 3\ell + 78 - \ell}{78}$

$\Rightarrow 6\ell = 5(156 - 4\ell)$

$\Rightarrow 26\ell = 5 \times 156$

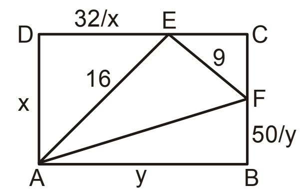
$\Rightarrow \ell = \frac{5 \times 156}{26} = 30$

But ℓ must be less than 26, hence no solution is possible.

25. Let ABCD be a rectangle and let E and F be points on CD and BC respectively such that area (ADE) = 16, area (CEF) = 9 and area (ABF) = 25. What is the area of triangle AEF ?

Soln. (30)

Let AD = x and AB = y



ar $(\Delta ADE) = 16 \quad \Rightarrow \frac{1}{2} \times x \times DE = 16$

$\Rightarrow DE = \frac{32}{x}$

Similarly, $BF = \frac{50}{y}$

$CE = AB - DE = y - \frac{32}{x}$ and $CF = BC - BF = x - \frac{50}{y}$

Now $ar(\Delta CEF) = 9$

$\Rightarrow \frac{1}{2} \times \left(y - \frac{32}{x}\right)\left(x - \frac{50}{y}\right) = 9$

$\Rightarrow xy + \frac{1600}{xy} - 32 - 50 = 18$

$\Rightarrow xy + \frac{1600}{xy} = 100$

$\Rightarrow A^2 - 100A + 1600 = 0 \qquad (A = xy)$

$\Rightarrow (A - 20)(A - 80) = 0$

$\Rightarrow A = 20 \text{ or } 80$

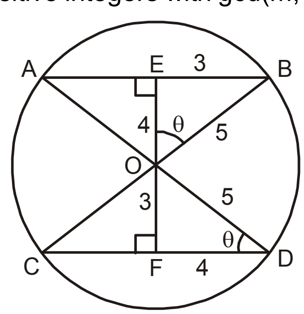
But A = xy = $ar(\square ABCD) > 20$

$\therefore A = 80$

$\therefore ar(\Delta AEF) = 80 - (16 + 25 + 9) = 30$

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose AB = 6, CD = 8. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n)/k$, where m, n, k are positive integers with gcd(m, n, k) = 1. What is the value of m + n + k?

Soln. (75)



Draw $OE \perp AB$ and $OF \perp CD$.

Clearly $EB = \frac{AB}{2} = 3$, $FD = \frac{CD}{2} = 4$

$OE = \sqrt{5^2 - 3^2} = 4$ and $OF = \sqrt{5^2 - 4^2} = 3$

$\therefore \Delta OEB \sim \Delta DFO$

Let $\angle EOB = \angle ODF = \theta$, then

$\angle BOD = \angle AOC = 180^\circ - (\theta + 90^\circ - \theta) = 90^\circ$

Now area of portion between the chords

$= 2 \times$ (area of minor sector BOD) $+2\times ar(\Delta AOB)$

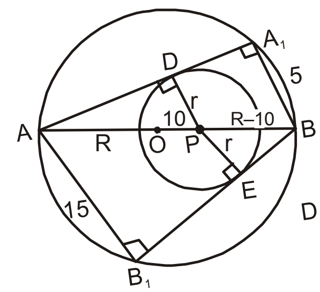
$$= 2\times\frac{\pi\times 5^2}{4}+2\times\frac{1}{2}\times 6\times 4 \;=\frac{25\pi}{2}+24=\frac{25\pi+48}{2}$$

$\therefore\ m=25, n=48$ and $k=2$

$\therefore\ m+n+k=75$

27. Let Ω_1 be a circle with centre O and let AB be a diameter of Ω_1. Le P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5$, $AB_1 = 15$ and OP = 10, find the radius of Ω_1.

Soln. (20)



Let radius of Ω_1 be R and that of Ω_2 be r

From figure, $\Delta ADP \sim \Delta AA_1B$

$$\Rightarrow \frac{DP}{A_1B}=\frac{AP}{AB}$$

$$\Rightarrow \frac{r}{5}=\frac{R+10}{2R} \qquad \ldots(1)$$

Again, $\Delta BPE \sim \Delta BAB_1$

$$\therefore \frac{PE}{AB_1}=\frac{BP}{BA}$$

$$\Rightarrow \frac{r}{15}=\frac{R-10}{2R} \qquad \ldots(2)$$

Dividing (1) by (2),

$$3=\frac{R+10}{R-10}\Rightarrow R=20$$

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of 3pq for **all** positive integers n. Find the least possible value of p + q.

Soln. (28)

Let $p\le q$.

For $p=2, n=2$

$$n^{3pq}-n=2^{6q}-2=(3-1)^{6q}-2 \equiv -1 (\text{mod } 3)$$

$\therefore\ p\ne 2$

$\therefore$ 3 pq is odd.

Let 3 pq = 2k + 1, then

$$n^{3pq}-n=n^{2k+1}-n=n((n^2)^k-1)$$

$\therefore\ n(n^2-1) \mid n^{3pq}-n$ and $3 \mid n(n^2-1)$

$\therefore\ 3 \mid n^{3pq}-n\ \forall n$

Now, $n^{3pq}-n=((n^{3q})^p-n^{3q})+(n^{3q}-n)$

But using fermat's lettle theorem,

$p \mid (n^{3q})^p - n^{3q}$

$\therefore \quad p \mid n^{3q} - n$

i.e. $p \mid n(n^{3q-1} - 1)\ \forall n$

but $p \mid n(n^{p-1} - 1)\ \forall n$

so, $p - 1 \mid 3q - 1$

Similarly $q - 1 \mid 3p - 1$.

The pair with least possible sum satisfying above conditions is (11, 17).

$\therefore$ least value of p + q = 11 + 17 = 28.

29. For each positive integer n, consider the highest common factor h_n of the two numbers n! + 1 and (n + 1)!. For n < 100, find the largest value of h_n.

Soln. (97)

$\because$ 2,3,......,n each divides $\underline{|n+1}$ but does not divide $\underline{|n}+1$

$\therefore$ any common factor of $\underline{|n}+1$ and $\underline{|n+1}$ must be a factor of n + 1 also.

But any factor of n + 1 other than 1 and itself does not divide $\underline{|n}+1$.

so either $\left(\underline{|n}+1, \underline{|n+1}\right) = 1$ or n + 1 is a prime.

But if n + 1 is prime, then by Wilson's theorem, $n+1 \mid \underline{|n}+1$ and so $\left(\underline{|n}+1, \underline{|n+1}\right) = n+1$.

$\therefore$ largest value of h_n = largest prime less than 100 = 97

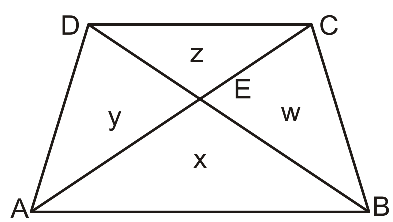
30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium ABCD. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

Soln. (13)

Let x, y, z, w be areas of the four triangles as shown in figure.

then $ar(\Delta ADB) = ar(\Delta ACB)$

$\Rightarrow x + y = x + w \Rightarrow y = w$



Also $\dfrac{AE}{EC} = \dfrac{ar(\Delta ADE)}{ar(\Delta DEF)} = \dfrac{ar(\Delta AEB)}{ar(\Delta BEC)}$

$\Rightarrow \dfrac{y}{z} = \dfrac{x}{w} = \dfrac{x}{y} \Rightarrow y^2 = zx$

$\Rightarrow z, y, x$ are in G.P.

Let y = zr and x = zr^2, where $r \geq 1$.

To make area of trapezium ABCD maximum, we take zy = z^2r = 576

and yw = z^2r^2 = 1296 $\quad (\because z \leq y \leq x)$

$\therefore \dfrac{z^2r^2}{z^2r} = \dfrac{1296}{576} \Rightarrow r = \dfrac{9}{4} \Rightarrow z = 16$

$\therefore$ area of trapezium ABCD

= x + y + z + w = $zr^2 + 2zr + z$

$= z(1+r)^2 = 16\left(1+\dfrac{9}{4}\right)^2 = 13^2$

$\therefore$ Answer is 13.

ANSWER - KEY

1.	(28)	2.	(73)	3.	(16)	4.	(25)	5.	(48)
6.	(83)	7.	(29)	8.	(7)	9.	(60)	10.	(48)
11.	(60)	12.	(42)	13.	(24)	14.	(10)	15.	(51)
16.	(36)	17.	(Bonus)	18.	(21)	19.	(36)	20.	(73)
21.	(60)	22.	(36)	23.	(47)	24.	(Bonus)	25.	(30)
26.	(75)	27.	(20)	28.	(28)	29.	(97)	30.	(13)