Regional Mathematical Olympiad-2017
Solutions

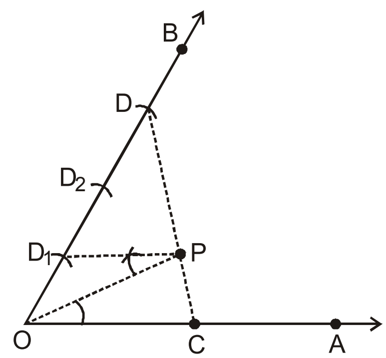
1. Let AOB be a given angle less then 180º and let P be an interior point of the angular region determined by $\angle$AOB. Show, with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB, and CP : PD = 1 : 2.

Sol. Construction :

We proceed as follows :

(i) First join CP using ruler.

(ii) Now we draw a line through P parallel to OA to meet OB at D_1. For this we draw $\angle OPD_1 = \angle POC$ using compasses.



(iii) Now using compasses we take points D_2 and D on OB such that $OD_1=D_1D_2=D_2D$, D_2 lying between D_1 and B and D lying between D_2 and B.

(iv) Join DP and produce it to meet OA at C, using ruler. CD is the required line segment.

Proof: $\because\ \angle OPD_1 = \angle POC$

$\therefore\ D_1P \,||\, OC$ ... (i)

$\because\ OD_1 = D_1D_2 = D_2D$

$\therefore\ \dfrac{OD_1}{D_1D} = \dfrac{1}{2}$

Now In Δ DOC, $D_1P \,||\, OC$

$\therefore$ by the Thales theorem, $\dfrac{CP}{PD} = \dfrac{D_1O}{D_1D} = \dfrac{1}{2}$. Q.E.D.

2. Show that the equation

$a^3 + (a+1)^3 + (a+2)^3 + (a+3)^3 + (a+4)^3 + (a+5)^3 + (a+6)^3 = b^4 + (b+1)^4$ has no solutions in integers a, b.

Sol. Let m = a + 3, $m \in I$, then from the given equation,

$$(m-3)^3 + (m-2)^3 + (m-1)^3 + m^3 + (m+1)^3 + (m+2)^3 + (m+3)^3 = b^4 + (b+1)^4$$

$\Rightarrow\ 7(m^3+12m)=b^4+(b+1)^4$

$\therefore\ 7\,|\,b^4+(b+1)^4$ ... (i)

Let b = 7k + r, where $k\in I$, and $r\in\{0,\pm1,\pm2,\pm3\}$.

$r=0 \text{ or } -1 \Rightarrow b^4+(b+1)^4=1(\text{mod } 7)$

r = 1 ro –2 $\Rightarrow b^4+(b+1)^4=3(\text{mod } 7)$

r = 2 ro – 3 $\Rightarrow b^4+(b+1)^4=6(\text{mod } 7)$

r = 3 $\Rightarrow b^4+(b+1)^4=1(\text{mod } 7)$

$\therefore\ 7\nmid b^4+(b+1)^4$

which is a contradiction to (1).

Hence the given equation has no integer solution.

3. Let $P(x)=x^2+\frac{1}{2}x+b$ and $Q(x)=x^2+cx+d$ be two polynomials with real coefficients such that $P(x)Q(x)=Q(P(x))$ for all real x. Find all the real roots of $P(Q(x))=0$.

Sol. Given $P(x)Q(x)=Q(P(x))\ \ \forall x\in R$

$$\Rightarrow (x^2+\frac{1}{2}x+b)(x^2+cx+d)=(x^2+\frac{1}{2}x+b)^2+c(x^2+\frac{1}{2}x+b)+d\quad \forall x\in R$$

Equating coefficients of x^3, x^2, x and constant terms respectively from both the sides, we get

$$c+\frac{1}{2}=1 \Rightarrow c=\frac{1}{2} \qquad \text{...(i)}$$

$$b+d+\frac{c}{2}=2b+\frac{1}{4}+c \Rightarrow d-b=\frac{1}{2} \qquad \text{...(ii)} \quad \text{(using (i))}$$

$$bc+\frac{d}{2}=b+\frac{c}{2} \Rightarrow d-b=\frac{1}{2}$$

$$bd=b^2+bc+d \Rightarrow b(b+\frac{1}{2})=b^2+\frac{b}{2}+b+\frac{1}{2} \quad \text{(using (i) \& (ii))}$$

$$\Rightarrow b=-\frac{1}{2} \qquad \text{... (iii)}$$

$$\Rightarrow d=0 \qquad \text{... (iv)} \qquad \text{(using (ii))}$$

$\therefore \quad P(x) = x^2 + \frac{1}{2}x - \frac{1}{2}$ and $Q(x) = x^2 + \frac{1}{2}x$

Now $P(x) = 0 \Rightarrow 2x^2 + x - 1 = 0 \Rightarrow x = -1$ or $\frac{1}{2}$

$\therefore \quad P(Q(x)) = 0 \Rightarrow Q(x) = -1$ or $\frac{1}{2}$

$\Rightarrow \quad x^2 + \frac{1}{2}x = -1$ or $x^2 + \frac{1}{2}x = \frac{1}{2}$

$\Rightarrow \quad 2x^2 + x + 2 = 0$ or $2x^2 + x - 1 = 0$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{-15}}{4}$ or $x = -1$ or $\frac{1}{2}$

$\therefore$ real roots of P(Q(x)) = 0 are –1 and $\frac{1}{2}$.

4. Consider n^2 unit squares in the xy-plane centred at point (i, j) with integer coordinates, $1 \le i \le n, 1 \le j \le n$. It is required to colour each unit square in such a way that when ever $1 \le i < j \le n$ and $1 \le k < l \le n$, the three squares with centres at (i, k), (j, k) (j, *l*) have distinct colours. What is the least possible number of colours needed ?

Sol. Given squares with centres at (i, k), (j, k), (j, l) have distinct colours for all $1 \le i < j \le n$ and $1 \le k < l \le n$.

In particular, for any given $k \in \{1, 2, ..., n-1\}$ squares with centres at (i, k) and (j, k) must have distinct colours for all $1 \le i < j \le n$. Which means in any row (except the top row) all squares have distinct colours.

Similarly, for any given $j \in \{2, 3, ..., n\}$ the squares with centres (j, k) and (j, l) must have distinct colours for all $1 \le k < l \le n$. Which means in any column (except the first) all squares have distinct colours.

Also squares with centres (i, k) and (j, l) have distinct colours for all $1 \le i < j \le n$ and $1 \le k < l \le n$. Which means any square to the upper right of a given square must have different colours. Thus all the 2n – 1 squares in bottom row and rightmost column must have distinct colours.

a_n	a_{n+1}				a_{2n-1}
	a_n	a_{n+1}			
		a_n	a_{n+1}		
a_3			a_n	a_{n+1}	
a_2	a_3			a_n	a_{n+1}
a_1	a_2	a_3			a_n

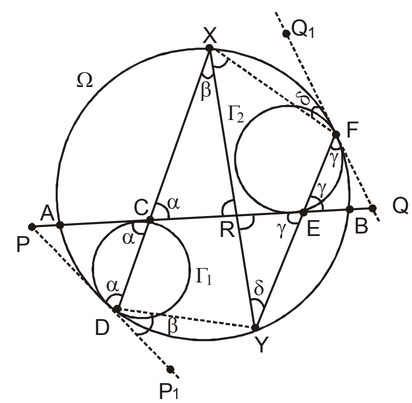
Suppose $a_1, a_2, ..., a_{2n-1}$ be 2n – 1 distinct colours, then we can colour the squares as shown in the figure. Any square with centre at (i, j) will have colour a_{i+j-1}. It is easy to see that the colouring satisfies the given conditions. Hence we need at least 2n – 1 colours.

5. Let Ω be a circle with a chord AB which is not a diameter. Let Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise, let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω.

Sol. Join DY and FX. Draw common tangent PP_1 to circles Ω and Γ_1 at D to meet BA at P and common tangent QQ_1 to circles Ω and Γ_2 at F to meet AB at Q. Let AB and XY meet at R. To prove XY is diameter of circle Ω, it is enough to show that $\angle XDY = 90^\circ$.

Let $\angle PDX = \alpha$, $\angle P_1DY = \beta$,

$\angle QFY = \gamma$ and $\angle Q_1FX = \delta$.



Using alternate segment theorem in circle Ω, we get

$\angle DXY = \angle P_1DY = \beta$ ... (i)

$\angle FYX = \angle Q_1FX = \delta$ ... (ii)

$\because$ PD and PC are tangents from P to the circle Γ_1

$\therefore$ PD = PC

$\Rightarrow \angle PCD = \angle PDC = \angle PDX = \alpha$

$\Rightarrow \angle XCR = \angle PCD = \alpha$ ... (iii)

$\therefore$ In ΔXCR, $\angle XRC = 180^\circ - (\angle CXR + \angle XCR)$

$= 180^\circ - (\alpha + \beta)$ ... (iv) $(\because \angle CXR = \angle DXY = \beta)$

Similarly in Δ ERY,

$\angle RYE = \angle FYX = \delta$ and $\angle REY = \angle FEQ = \angle QFE = \gamma$

$\therefore \angle ERY = 180^\circ - (\gamma + \delta)$ ... (v)

But $\angle XRC = \angle ERY$

$\therefore$ From (iv) & (v), $\alpha + \beta = \gamma + \delta$ ... (vi)

Now DXFY is a cyclic quadrilateral

$\therefore \angle XDY + \angle XFY = 180^\circ$

$\Rightarrow 180^\circ - (\alpha + \beta) + 180^\circ - (\gamma + \delta) = 180^\circ$

$\Rightarrow \alpha + \beta + \gamma + \delta = 180^\circ$

$\Rightarrow 2(\alpha + \beta) = 180^\circ$ (using (vi))

$\Rightarrow \alpha + \beta = 90^\circ$

$\therefore \angle XDY = 180^\circ - (\alpha + \beta) = 90^\circ$. Q.E.D.

6. Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1}\le\frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1}.$$

Sol. We have to prove

$$\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1}\le\frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1} \qquad \text{... (i)}$$

i.e. $\left(\frac{x-1}{y-1}-\frac{x+1}{y+1}\right)+\left(\frac{y-1}{z-1}-\frac{y+1}{z+1}\right)+\left(\frac{z-1}{x-1}-\frac{z+1}{x+1}\right)\ge 0$

i.e. $\frac{2(x-y)}{y^2-1}+\frac{2(y-z)}{z^2-1}+\frac{2(z-x)}{x^2-1}\ge 0$

i.e. $\frac{x}{y^2-1}+\frac{y}{z^2-1}+\frac{z}{x^2-1}\ge\frac{y}{y^2-1}+\frac{z}{z^2-1}+\frac{x}{x^2-1}$ ... (ii)

Now to prove (ii), we will use the rearrangement inequality on the two sequences x, y, z and $\frac{1}{x^2-1},\frac{1}{y^2-1},\frac{1}{z^2-1}$.

$\because$ $x, y, z > 1$

$\therefore$ $\frac{1}{x^2-1},\frac{1}{y^2-1},\frac{1}{z^2-1}>0$

Let $x \ge y \ge z$, then $\frac{1}{x^2-1}\le\frac{1}{y^2-1}\le\frac{1}{z^2-1}$

$\therefore$ by rearranging inequality

$$x.\frac{1}{x^2-1}+y.\frac{1}{y^2-1}+z.\frac{1}{z^2-1}\le x.\frac{1}{y^2-1}+y.\frac{1}{z^2-1}+z.\frac{1}{z^2-1}$$

Which proves (ii).

The same argument applies for any order of x, y and z. Hence the result.