JEE (ADVANCED) 2019 PAPER-1

[PAPER WITH SOLUTION]

HELD ON SUNDAY 27TH MAY, 2019

MATHEMATICS

SECTION 1 (Maximum Marks : 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : **+3** If ONLY the correct option is chosen.

Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : **–1** In all other cases.

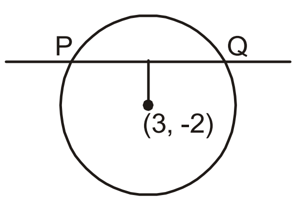
1. A line $y = mx + 1$ intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the point P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct?

(1) $4 \le m < 6$ (2) $6 \le m < 8$ (3) $-3 \le m < -1$ (4) $2 \le m < 4$

Ans. (4)

Sol. Let midpoint of PQ is $\left(-\frac{3}{5}, 1-\frac{3m}{5}\right)$

Slope of PQ



$$m = \frac{18}{15-3m}$$

$$\Rightarrow m = \frac{6}{5-m}$$

$$\Rightarrow m^2 - 5m + 6 = 0$$

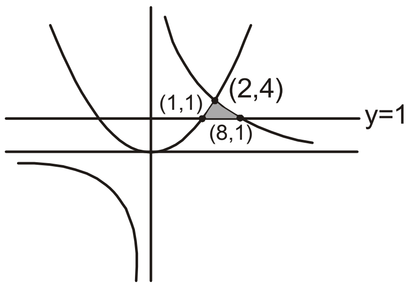
$$\therefore m = 2, 3$$

2. The area of the region $\{(x, y): xy \le 8,\ 1 \le y \le x^2\}$ is

(1) $16\log_e 2 - 6$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $8\log_e 2 - \frac{14}{3}$ (4) $8\log_e 2 - \frac{7}{3}$

Ans. (2)

Sol.



$$= \int_1^2 (x^2 - 1)dx + \int_2^8 \left(\frac{8}{x} - 1\right)dx$$

$$= \left|\frac{x^3}{3} - x\right|_1^2 + \left|8\log x - x\right|_2^8$$

$$= \frac{-14}{3} + 16\log 2$$

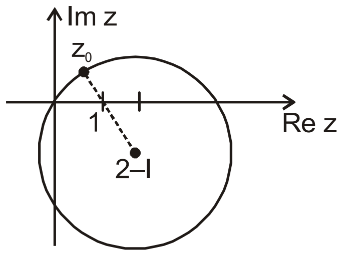
3. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|}; z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is

(1) $\frac{3\pi}{4}$ (2) $\frac{\pi}{2}$ (3) $-\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

Ans. (3)

Sol. z_0 is the complex number satisfying $|z - (2 - i)| \ge \sqrt{5}$ which is at least distance from 1.

Let $z_0 = x_0 + iy_0$, if will be the complex number as shown in the figure.



Now, $\dfrac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i} = \dfrac{4-2x_0}{2iy_0+2i} = -\left(\dfrac{2-x_0}{y_0+1}\right)i$

$\therefore$ Principal argument of $\dfrac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is $\dfrac{-\pi}{2}$

4. Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$,

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers and I is the 2 × 2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

Then the value of $\alpha^* + \beta^*$ is

(1) $-\dfrac{29}{16}$ (2) $-\dfrac{31}{16}$ (3) $-\dfrac{37}{16}$ (4) $-\dfrac{17}{16}$

Ans. (1)

Sol. Determinant of $M = |M| = \sin^4\theta\cos^4\theta + (1+\sin^2\theta)(1+\cos^2\theta)$

$$= 2 + \sin^2\theta\cos^2\theta + \sin^4\theta\cos^4\theta$$

$$M^{-1} = \frac{1}{|M|}\text{adj}\, M = \frac{1}{|M|}\begin{bmatrix} \cos^4\theta & 1+\sin^2\theta \\ -(1+\cos^2\theta) & \sin^4\theta \end{bmatrix}$$

Now, $\alpha I + \beta M^{-1} = \begin{bmatrix} \alpha + \dfrac{\beta}{|M|}\cos^4\theta & \dfrac{\beta}{|M|}(1+\sin^2\theta) \\ -\dfrac{\beta}{|M|}(1+\cos^2\theta) & \alpha + \dfrac{\beta}{|M|}\sin^4\theta \end{bmatrix}$

For $\alpha I + \beta M^{-1} = M$, we must have

$$\alpha + \frac{\beta}{|M|}\cos^4\theta = \sin^4\theta$$

$$\alpha + \frac{\beta}{|M|}\sin^4\theta = \cos^4\theta$$

$$\frac{\beta}{|M|}(1+\sin^2\theta) = -(1+\sin^2\theta)$$

And $-\frac{\beta}{|M|}(1+\cos^2\theta) = 1+\cos^2\theta$

Therefore, $\beta = -|M|$ & $\alpha = \cos^4\theta + \sin^4\theta$

as $\alpha = \cos^4\theta + \sin^4\theta = (\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$$\alpha = 1 - \frac{1}{2}\sin^2 2\theta$$

$\therefore$ $\alpha^* = \text{minimum of } \alpha = \frac{1}{2}$

and $\beta = -|M|$

$$= -(2 + \sin^2\theta\cos^2\theta + \sin^4\theta\cos^4\theta)$$

$$= -\left(\left(\sin^2\theta\cos^2\theta + \frac{1}{2}\right)^2 + \frac{7}{4}\right)$$

$$= -\left(\left(\frac{1}{4}\sin^2 2\theta + \frac{1}{2}\right)^2 + \frac{7}{4}\right)$$

$\therefore$ $\beta^* = \text{minimum of } \beta = -\frac{37}{16}$

$\therefore$ $\alpha^* + \beta^* = -\frac{29}{16}$

SECTION 2 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct option(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : **+4** If only (all) the correct option(s) is(are) chosen.
  Partial Marks : **+3** If all the four options are correct but ONLY three options are chosen.
  Partial Marks : **+2** If three or more options are correct but ONLY two options are chosen and both of which are correct.
  Partial Marks : **+1** If two or more options are correct but ONLY one option is chosen and it is a correct option.
  Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).
  Negative Marks : **–1** In all other cases.
- **For example :** In a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answer, then

  choosing ONLY (A), (B) and (D) will get +4 marks.
  choosing ONLY (A) and (B) will get +2 marks.
  choosing ONLY (A) and (D) will get +2 marks.
  choosing ONLY (B) and (D) will get +2 marks.
  choosing ONLY (A) will get +1 mark.
  choosing ONLY (B) will get +1 mark.
  choosing ONLY (D) will get +1 mark.
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get –1 mark.

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1. Let α and β be the roots of $x^2 - x - 1 = 0$ with $\alpha > \beta$. For all positive integers n, define

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1$$

$$b_1 = 1 \text{ and } b_n = a_{n-1} + a_{n+1}, \quad n \geq 2$$

Then which of the following options is/are correct?

(1) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(2) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(3) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

(4) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Ans. (1), (2), (3)

Sol. (1) $\sum_{n=1}^{\infty}\frac{a_n}{10^n}=\sum_{n=1}^{\infty}\frac{\alpha^n-\beta^n}{(\alpha-\beta)10^n}\quad\left(\because a_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$

$$=\frac{1}{\alpha-\beta}\left(\frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}}-\frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}\right)$$

$$=\frac{1}{\alpha-\beta}\left(\frac{\alpha}{10-\alpha}-\frac{\beta}{10-\beta}\right)$$

$$=\frac{1}{\alpha-\beta}\left(\frac{10\alpha-\alpha\beta-10\beta+\alpha\beta}{(10-\alpha)(10-\beta)}\right)$$

$$=\frac{1}{\alpha-\beta}\left(\frac{10(\alpha-\beta)}{100-10(\alpha+\beta)+\alpha\beta}\right)$$

$$=\frac{10}{100-10-1}=\frac{10}{89}$$

(2) $a_1+a_2+.....+a_n=\sum a_i$

$$=\frac{\sum\alpha^i-\sum\beta^i}{\alpha-\beta}$$

$$=\frac{\frac{\alpha(1-\alpha^n)}{1-\alpha}-\frac{\beta(1-\beta^n)}{1-\beta}}{\alpha-\beta}$$

$$=\frac{(\alpha+1)(1-\alpha^n)-(\beta+1)(1-\beta^n)}{(1-\alpha)(\alpha-\beta)(1-\beta)}$$

$$=\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}-1$$

$$=-1+a_{n+2}=a_{n+2}-1$$

(3) $b_n=a_{n-1}+a_{n+1}=\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}+\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$

$$=\frac{\alpha^{n-1}\left(\alpha^2+1\right)-\beta^{n-1}\left(\beta^2+1\right)}{\alpha-\beta}$$

$$=\frac{\alpha^{n-1}(\alpha+2)-\beta^{n-1}(\beta+2)}{\alpha-\beta}$$

$$= \frac{\alpha^{n-1}\left(\frac{5+\sqrt{5}}{2}\right) - \beta^{n-1}\left(\frac{5-\sqrt{5}}{2}\right)}{\alpha - \beta}$$

$$= \frac{\sqrt{5}\left(\alpha^n + \beta^n\right)}{\alpha - \beta} = \alpha^n + \beta^n$$

(4) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \sum\left(\frac{\alpha}{10}\right)^n + \sum\left(\frac{\beta}{10}\right)^n$

$$= \frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}} = \frac{\alpha}{10-\alpha} + \frac{\beta}{10-\beta}$$

$$= \frac{10(\alpha+\beta) - 2\alpha\beta}{100 - 10(\alpha+\beta) + \alpha\beta} = \frac{10+2}{89} = \frac{12}{89}$$

2. Let L_1 and L_2 denote the lines

$\vec{r} = \hat{i} + \lambda(-\hat{i} + 2\hat{j} + 2\hat{k}),\ \lambda \in R$ and

$\vec{r} = \mu(2\hat{i} - \hat{j} + 2\hat{k}),\ \mu \in R$

respectively, If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

(1) $\vec{r} = \frac{1}{3}(2\hat{i} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})\ t \in R$

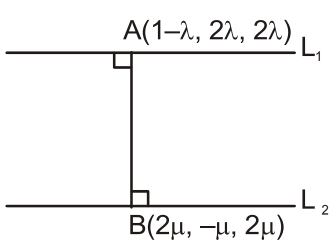
(2) $\vec{r} = t(2\hat{i} + 2\hat{j} - \hat{k}),\ t \in R$

(3) $\vec{r} = \frac{2}{9}(2\hat{i} - \hat{j} + 2\hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})\ t \in R$

(4) $\vec{r} = \frac{2}{9}(4\hat{i} + \hat{j} + \hat{k}) + t(2\hat{i} + 2\hat{j} - \hat{k})\ t \in R$

Ans. (1),(3),(4)

Sol.



Both given lines are skew lines.

So direction ratio of any line perpendicular to these lines are (2, 2, -1)

If points at shortest distance between given lines are $A(1-\lambda, 2\lambda, 2\lambda)$ and $B(2\mu, -\mu, 2\mu)$.

Then $\overrightarrow{AB} \perp \text{Line } L_1$

$\overrightarrow{AB} \perp \text{Line } L_2$

So, $A\left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right)$

Now, equation of required lines $\vec{r} = \left(\frac{8}{9}\hat{i} + \frac{2}{9}\hat{j} + \frac{2}{9}\hat{k}\right) + \alpha\left(2\hat{i} + 2\hat{j} - \hat{k}\right)$

Hence 1, 3, 4 are correct options.

3. Let $f : R \to R$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3. \end{cases}$$

Then which of the following options is/are correct?

(1) f is increasing on $(-\infty, 0)$

(2) f' has a local maximum at x = 1

(3) f' is NOT differentiable at x = 1

(4) f is onto

Ans. (2,3,4)

Sol. $$f'(x) = \begin{cases} 5x^4 + 20x^3 + 30x^2 + 20x + 3, & x < 0 \\ 2x - 1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \\ \log_e(x-2), & x \ge 3 \end{cases}$$

$$f'(x) = \begin{cases} 5(x+1)^4 - 2, & x < 0 \\ 2x - 1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \\ \log_e(x-2), & x \ge 3 \end{cases}$$

$$f'(x) = \begin{cases} 5(x+1)^4 - 2, & x < 0 \\ 2x - 1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \\ \log_e(x-2), & x \ge 3 \end{cases}$$

(1) For increasing in x < 0

Put $5(x+1)^4 - 2 > 0 \quad \Rightarrow (x+1)^4 > \frac{2}{5}$

Which is not true $\forall$ x < 0

So, (1) is not correct.

(2) For $f'(x)$ has local maximum at x = 1

$$f'(x) = \begin{cases} 2x-1, & 0 \le x < 1 \\ 2x^2 - 8x + 7, & 1 \le x < 3 \end{cases}$$

$$f''(x) = \begin{cases} 2, & 0 \le x < 1 \\ 4x - 8, & 1 \le x < 3 \end{cases}$$

Sign of equation $f''(x)$ changes from (+ve) to negative as x increases from (1 – h) to (1 + h) where $h \to 0$

So $f'(x)$ has local maximum at x = 1.

So (2) is correct.

(3) $f'(x)$ is not differentiable at x = 1

R. H. D at x = 1 is 2

L. H. D at x = 1 is -4

So (3) is correct.

(4) Range of $x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1$ is super set of set $(-\infty, 1)$

Range of $(x-2)\log_e(x-2) - x + \frac{10}{3}$ is $\left[\frac{1}{3}, \infty\right)$

So range of f(x) is R

$\therefore$ (4) is correct.

4. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and $\operatorname{adj} M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

where a and b are real numbers. Which of the following options is/are correct?

(1) a + b = 3

(2) $(\operatorname{adj} M)^{-1} + \operatorname{adj} M^{-1} = -M$

(3) If $M\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

(4) $\det(\operatorname{adj} M^2) = 81$

Ans. (1,2,3)

Sol. $(\text{adj}\,M)_{11} = 2-3b$ and $(\text{adj}\,M)_{22} = -3a$

Equate them to

$$2-3b = -1 \text{ and } -3a = -6$$

$$3b = 3 \quad \text{and} \quad a = 2$$

$$b = 1$$

$$a+b = 3$$

$$\det(M) = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = -2$$

$$M^{-1} = \frac{\text{adj}\,(M)}{|M|}, \qquad |M| = \det(M)$$

$$\text{adj}(M) = |M|M^{-1}$$

$$\text{adj}(M) = -2M^{-1}$$

$$(\text{adj }M)^{-1} = -\frac{1}{2}M \qquad \ldots\text{(i)}$$

Also $\left(\text{adj }M^{-1}\right) = \left(M^{-1}\right)^{-1}\det\left(M^{-1}\right) = \dfrac{M}{-2}$

$$\left(\text{adj }M^{-1}\right) = \frac{M}{-2} \qquad \ldots\text{(ii)}$$

Adding (i) and (ii) we get

$$\left(\text{adj }M^{-1}\right) + \left(\text{adj }M^{-1}\right) = -M$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\beta + 2\gamma = 1$$

$$\alpha + 2\beta + 3\gamma = 2$$

$$3\alpha + \beta + \gamma = 3$$

Solving we get $\alpha = 1,\ \beta = -1,\ \gamma = 1$

So $\alpha - \beta + \gamma = 3$

$$\det(\text{adj}\,M^2) = 16$$

5. In a non-right-angled triangle ΔPQR, let p,q,r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p = \sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct?

(1) Length of RS = $\frac{\sqrt{7}}{2}$

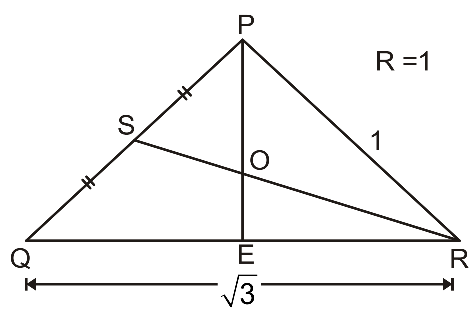
(2) Length of OE = $\frac{1}{6}$

(3) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}\left(2-\sqrt{3}\right)$

(4) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$

Ans. (1,2,3)

Sol.



From sine Rule,

$$\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2}$$

$$\sin P = \frac{\sqrt{3}}{2} \quad \sin Q = \frac{1}{2}$$

$$P = 60^\circ \text{ or } 120^\circ, \quad Q = 30^\circ, \ 150^\circ$$

So, $P = 120^\circ, \ Q = 30^\circ$ & $R = 30^\circ$ is possible.

So, Δ is isosceles hence PE is median

So, PQ = 1,

(1) RS = $\frac{1}{2}\sqrt{2+2\times3-1} = \frac{\sqrt{7}}{2}$ (correct)

(2) $OE = \frac{1}{3}PE = \frac{1}{6}$ (correct)

(3) $r = \frac{\Delta}{S} = \frac{\frac{1}{2}\left(1 \times \sqrt{3} \times \frac{1}{2}\right)}{\frac{2+\sqrt{3}}{2}}$ (correct)

(4) Area Δ SOE $= \frac{\Delta}{4} \times \frac{1}{3} = \frac{\sqrt{3}}{4} \times \frac{1}{12}$ (incorrect)

6. Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct?

(1) $xy' + \sqrt{1-x^2} = 0$

(2) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(3) $xy' - \sqrt{1-x^2} = 0$

(4) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

Ans. (1,4)

Sol. Suppose $(a, f(a))$ is a point on Γ

Now, equation of tangent.

$y - f(a) = f'(a)(x-a)$

Put x = 0

$y - f(a) = -af'(a)$

$y = f(a) - af'(a)$

$y_P = (0, f(a) - af'(a))$

$P_{y_P} = \sqrt{a^2 + (af'(a))^2} = 1$

$a^2 + a^2(f'(a))^2 = 1$

$(f'(a))^2 = \frac{1-a^2}{a^2} \Rightarrow y' = -\frac{\sqrt{1-x^2}}{x} = xy' + \sqrt{1-x^2} = 0, \quad (\because y' < 0 \ \forall x \in (0,1))$

[(1) is correct]

$$\int (f'(x))dx = -\int \sqrt{\frac{1-x^2}{x^2}}dx$$

Put $\sqrt{1-x^2} = t$

$$\Rightarrow \; y = -\int \frac{t^2}{1-t^2}dt = -\left(\sqrt{1-x^2} - \ln\frac{1+\sqrt{1-x^2}}{x}\right) + C$$

Put $x = 1$ and y = 0 $\Rightarrow$ $C = 0$

[(4) is correct]

7. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}, \frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct?

(1) Probability that the chosen ball is green equals $\frac{39}{80}$

(2) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(3) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(4) Probability that the selected bag is B_3, given that the chosen ball is green, equals $\frac{5}{13}$

Ans. (1,2)

Sol.

	Bag 1	Bag 2	Bag 3
Red balls	5	3	5
Green balls	5	5	3
Total	10	8	8

Since $P(B_1) = \frac{3}{10},\; P(B_2) = \frac{3}{10},\; P(B_3) = \frac{4}{10}$

Suppose G is the event of choosing green ball from the selected bag

Now

(1) $P(G) = P(B_1)\cdot P\left(G/_{B_1}\right) + P(B_2)\cdot P\left(G/_{B_2}\right) + P(B_3)\cdot P\left(G/_{B_3}\right)$

$$= \left(\frac{3}{10}\times\frac{5}{10}\right)+\left(\frac{3}{10}\times\frac{5}{8}\right)+\left(\frac{4}{10}\times\frac{3}{8}\right)=\frac{39}{80}$$

(1) is correct answer.

(2) $P\left(\frac{\text{Ball chosen is green}}{\text{Ball is from bag } B_3}\right) = \frac{3}{8}$

(2) is correct answer**.**

(3) $P(B_3 \cap G) = P(B_3)\times P\left(G/_{B_3}\right) = \frac{4}{10}\times\frac{3}{8} = \frac{3}{20}$

(3) is incorrect answer.

(4) $$P\left(\frac{B_3}{G}\right) = \frac{P(B_3)\ P\left(G/_{B_3}\right)}{P(B_1)\cdot P\left(G/_{B_1}\right)+P(B_2)\cdot P\left(G/_{B_2}\right)+P(B_3)\cdot P\left(G/_{B_3}\right)}$$

$$= \frac{\frac{4}{10}\times\frac{3}{8}}{\left(\frac{3}{10}\times\frac{5}{10}\right)+\left(\frac{3}{10}\times\frac{5}{8}\right)+\left(\frac{4}{10}\times\frac{3}{8}\right)} = \frac{4}{13}$$

(4) is incorrect answer.

8. Define the collection $\{E_1, E_2, E_3, ...\}$ of ellipse and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows:

$E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1;$

R_1: rectangle of largest area, with sides parallel to the axes, inscribed in E_1;

E_n: ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of largest area inscribed in R_{n-1}, n > 1;

R_n: rectangle of largest area, with sides parallel to the axes, inscribed in E_n, n > 1

Then which of the following options is/are correct?

(1) $\sum_{n=1}^{N}(\text{area of } R_n) < 24,$ for each positive integer N

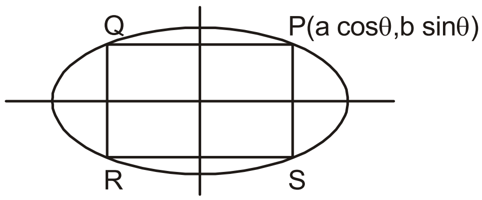
(2) The length of latus rectum of E_9 is $\frac{1}{6}$

(3) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(4) The eccentricities of E_{18} and E_{19} are NOT equal

Ans. (1,2)

Sol.



Area of rectangle is ellipse $= 2a\cos\theta \cdot 2b\sin\theta \quad \Rightarrow 2ab\sin(2\theta) \le 2ab$

Since E_1 is $\frac{x^2}{9}+\frac{y^2}{4}=1$ hence $R_1 = 2\cdot(3)\cdot(2) \; = 12$

Since, For E_1, a = 3, b = 2 hence $R_1 = 2(3)(2) = 12$

For E_2, $a=\frac{3}{\sqrt{2}},\; b=\frac{2}{\sqrt{2}}$ $\quad R_2 = 2\left(\frac{3}{\sqrt{2}}\right)\left(\frac{2}{\sqrt{2}}\right)=6$

For E_3, $a=\frac{3}{\left(\sqrt{2}\right)^2},\; b=\frac{2}{\left(\sqrt{2}\right)^2}$ $\quad R_3 = 2\left(\frac{3}{\left(\sqrt{2}\right)^2}\right)\left(\frac{2}{\left(\sqrt{2}\right)^2}\right)=3$

For E_4, $a=\frac{3}{\left(\sqrt{2}\right)^3},\; b=\frac{2}{\left(\sqrt{2}\right)^3}$ $\quad R_4 = 2\left(\frac{3}{\left(\sqrt{2}\right)^3}\right)\left(\frac{2}{\left(\sqrt{2}\right)^3}\right)=\frac{3}{2}$

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For E_n, $a=\frac{3}{\left(\sqrt{2}\right)^{n-1}},\; b=\frac{2}{\left(\sqrt{2}\right)^{n-1}}$ $\quad R_n = 2\left(\frac{3}{\left(\sqrt{2}\right)^{n-1}}\right)\left(\frac{2}{\left(\sqrt{2}\right)^{n-1}}\right)=\frac{3}{2^{n-3}}$

Now, (1) $\sum_{n=1}^{N}\left(\text{area of } R_n\right) = R_1 + R_2 + R_3 + ... + R_N$

$\Rightarrow\ 12+6+3+\frac{3}{2}+\frac{3}{4}...$ upto N terms $=\dfrac{12\left(1-\left(\frac{1}{2}\right)^{N}\right)}{\left(1-\frac{1}{2}\right)}$

$\Rightarrow\ 24\left(1-\frac{1}{(2)^{N}}\right)<24$ for each positive integer N.

Hence (1) is correct.

(2) The length of Latus Rectum of E_9 $=\dfrac{2b^2}{a}=\dfrac{2\cdot\left(\frac{2}{\left(\sqrt{2}\right)^8}\right)^2}{\frac{3}{\left(\sqrt{2}\right)^8}}=\dfrac{8}{(16)^2}\times\dfrac{16}{3}=\dfrac{1}{6}$

Hence (2) is correct.

(3) eccentricities of E_9 $(e)=\sqrt{1-\frac{b^2}{a^2}}\ \Rightarrow\ e=\sqrt{1-\dfrac{\left(\frac{2}{\left(\sqrt{2}\right)^8}\right)}{\left(\frac{3}{\left(\sqrt{2}\right)^8}\right)^2}}=\dfrac{\sqrt{5}}{3}$

Here distance of a focus from the centre in E_9 $\Rightarrow ae=\dfrac{3}{\left(\sqrt{2}\right)^8}\cdot\dfrac{\sqrt{5}}{3}=\dfrac{\sqrt{5}}{16}$

Hence (3) is incorrect answer.

(4) Here, eccentricity of all ellipses are equal.

Hence (4) is incorrect answer.

SECTION 3 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value of to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : **+3** If ONLY the correct numerical value is entered.

  Zero Marks **: 0** In all other cases.

1. If

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{\left(1+e^{\sin x}\right)\left(2-\cos 2x\right)}$$

Then $27I^2$ equals

Ans. 4

Sol.
$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{\left(1+e^{\sin x}\right)\left(2-\cos 2x\right)}$$

$$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{e^{\sin x}dx}{\left(1+e^{\sin x}\right)\left(2-\cos 2x\right)}$$

$$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{2-\cos 2x}$$

$$I = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x\, dx}{3\sec^2 x - 2} = \frac{2}{\pi}\int_{0}^{1} \frac{dt}{3t^2+1} \quad \text{(where t = tanx)}$$

$$I = \frac{2}{3\sqrt{3}} \quad : \quad 27I^2 = 04$$

2. Let S be the sample space of all 3 × 3 matrices with entries from the set $\{0, 1\}$. Let the events E_1 and E_2 be given by

$$E_1 = \{A \in S \ : \det A = 0\} \text{ and}$$

$$E_2 = \{A \in S \ : \text{sum of entries of A is } 7\}$$

If a matrix is chosen at random from S, then the conditional probility $P(E_1 | E_2)$ equals

Ans. (0.50)

Sol. E_2 : Sum of elements of A = 7

$\Rightarrow$ There are 7 one and 2 zero.

No. of such matrices $= \frac{9!}{7!2!} = {}^9C_2 = 36$

Out of all such matrices: E_1 will be those when both zero lie in the same row or in the same column.

i.e. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$

$$P\left(E_1/E_2\right) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{2 \times {}^3C_2 \times {}^3C_2}{36} = \frac{1}{2}$$

3. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\left\{\left|a + b\omega + c\omega^2\right|^2 : \text{a, b, c distinct non-zero integers}\right\}$$

equals

Ans. (3)

Sol. $\left|a + b\omega + c\omega^2\right|^2 = \left(a + b\omega + c\omega^2\right)\left(\overline{a + b\omega + c\omega^2}\right)$

$$= \left(a + b\omega + c\omega^2\right)\left(a + b\omega^2 + c\omega\right)$$

$$= a^2 + b^2 + c^2 - ab - bc - ca$$

$$= \frac{1}{2}\left\{(a-b)^2 + (b-c)^2 + (c-a)^2\right\}$$

Which is minimum when a, b, c are consecutive integers and so the minimum value

$$= \frac{1}{2}\left\{1^2 + 1^2 + 2^2\right\} = 3$$

4. Let $AP(a; d)$ denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If

$$AP(1;\ 3) \cap AP(2;\ 5) \cap AP(3;\ 7) = AP(a; d)$$

then $a + d$ equals

Ans. (157)

Sol. Let a_p, b_q and c_r be the p^{th}, q^{th} and r^{th} terms of AP(1; 3), AP(2; 5) and AP(3; 7) respectively such that

$$a_p = b_q = c_r$$

Then $1 + (p-1)\cdot 3 = 2 + (q-1)\cdot 5 = 3 + (r-1)\cdot 7$

$\Rightarrow$ $3p - 2 = 5q - 3 = 7r - 4$

$\Rightarrow$ $3p + 2 = 5q + 1 = 7r = \lambda$ (say)

$\therefore$ $p = \dfrac{\lambda - 2}{3},\ q = \dfrac{\lambda - 1}{5},\ r = \dfrac{\lambda}{7}$

Least value of λ for which p, q, r are positive integers is 56.

$\therefore$ $\lambda = \text{LCM}(3,5,7)\cdot k + 56 = 105k + 56, k \in W$

$\therefore$ $a_p = \lambda - 4 = 105k + 52$

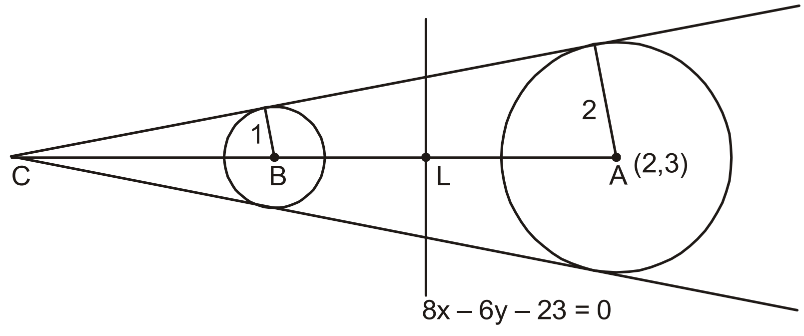
$\therefore$ $a = 52$ and $d = 105$

$\therefore$ $a + d = 157$

5. Let the point B be the reflection of the point A(2,3) with respect to the line 8x – 6y – 23 = 0. Let T_A & T_B be the circles of radii 2 & 1 with centres A & B respectively. Let T be the common tangent to the circles T_A & T_B such that both circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B then the length of the line segment AC is

Ans. (10)

Sol.



$\because$ $AB = 5$

$$AL = \frac{|8\times 2 - 6\times 3 - 23|}{\sqrt{8^2+(-6)^2}} = \frac{5}{2}$$

$\because \ \frac{CB}{CA} = \frac{1}{2} \quad \Rightarrow \quad \frac{CA-5}{CA} = \frac{1}{2}$

$\therefore \ CA = 10$

6. Three lines are given by

$\vec{r} = \lambda\hat{i}, \ \lambda \in R$

$\vec{r} = \mu(\hat{i}+\hat{j}), \ \mu \in R$

$\vec{r} = \nu(\hat{i}+\hat{j}+\hat{k}), \ \nu \in R$

Let the lines cut the plane x + y + z = 1 at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals

Ans. (0.75)

Sol. Put $(\lambda,0,0)$ in $x+y+z=1$

$\therefore \ \lambda+0+0=1 \qquad \therefore \ \lambda=1 \qquad \therefore \ P \equiv (1,0,0)$

Put $(\mu,\mu,0)$ in x + y + z = 1

$\Rightarrow \ 2\mu+0=1 \qquad \therefore \ \mu = \frac{1}{2} \qquad \therefore \ Q \equiv \left(\frac{1}{2},\frac{1}{2},0\right)$

Put (ν,ν,ν) in x + y + z = 1

$\Rightarrow \ 3\nu=1 \qquad \therefore \ \nu = \frac{1}{3} \qquad \therefore \ R = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

$\because$ Area of the $\Delta PQR = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|$

$$= \frac{1}{2}\left|\left(\frac{\hat{i}-\hat{j}}{2}\right)\times\left(\frac{2\hat{i}-\hat{j}-\hat{k}}{3}\right)\right|$$

$$= \frac{1}{12}\left|\hat{i}+\hat{j}+\hat{k}\right| = \frac{\sqrt{3}}{12}$$

$\therefore \ (6\Delta)^2 = \left(6\times\frac{\sqrt{3}}{12}\right)^2 = \frac{3}{4} = 0.75$