JEE (ADVANCED) 2019 PAPER-2

[PAPER WITH SOLUTION]

HELD ON SUNDAY 27TH MAY, 2019

MATHEMATICS

SECTION 1 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct option(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : **+4** If only (all) the correct option(s) is(are) chosen.
  Partial Marks : **+3** If all the four options are correct but ONLY three options are chosen.
  Partial Marks : **+2** If three or more options are correct but ONLY two options are chosen and both of which are correct.
  Partial Marks : **+1** If two or more options are correct but ONLY one option is chosen and it is a correct option.
  Zero Marks : **0** If none of the options is chosen (i.e. the question is unanswered).
  Negative Marks : **–1** In all other cases.
- **For example :** In a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answer, then
  choosing ONLY (A), (B) and (D) will get +4 marks.
  choosing ONLY (A) and (B) will get +2 marks.
  choosing ONLY (A) and (D) will get +2 marks.
  choosing ONLY (B) and (D) will get +2 marks.
  choosing ONLY (A) will get +1 mark.
  choosing ONLY (B) will get +1 mark.
  choosing ONLY (D) will get +1 mark.
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get –1 mark.

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1. For non-negative integers n, let

$$f(n)=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming $\cos^{-1}x$ takes values in $[0,\pi]$, which of the following options is/are correct ?

(1) If $\alpha = \tan(\cos^{-1} f(6))$, then $\alpha^2 + 2\alpha - 1 = 0$

(2) $f(4) = \frac{\sqrt{3}}{2}$

(3) $\sin(7\cos^{-1} f(5)) = 0$

(4) $\lim_{n\to\infty} f(n) = \frac{1}{2}$

Ans. (1, 2, 3)

Sol. $$f(n) = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{n+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

$$= \frac{\sum_{k=0}^{n} \cos\frac{\pi}{n+2} - \cos\left(\frac{2k+3}{n+2}\pi\right)}{\sum_{k=0}^{n} 1 - \cos\left(\frac{2k+2}{n+2}\pi\right)}$$

$$= \frac{(n+1)\cos\frac{\pi}{n+2} - \frac{\cos\left(\frac{n+3}{n+2}\pi\right)\sin\left(\frac{n+1}{n+2}\pi\right)}{\sin\left(\frac{\pi}{n+2}\right)}}{(n+1) - \frac{\cos\pi\sin\left(\frac{n+1}{n+2}\pi\right)}{\sin\left(\frac{\pi}{n+2}\right)}}$$

$$= \frac{(x+1)\cos\frac{\pi}{n+2} - \cos\left(\frac{n+3}{n+2}\pi\right)}{n+1+1}$$

$$= \frac{(n+1)\cos\left(\frac{\pi}{n+2}\right) + \cos\left(\frac{\pi}{n+2}\right)}{n+2} = \cos\left(\frac{\pi}{n+2}\right)$$

$\therefore\ f(x) = \cos\left(\frac{\pi}{n+2}\right)$ $\qquad\therefore\quad f(6) = \cos\frac{\pi}{8}$

$\therefore\ \cos^{-1} f(6) = \frac{\pi}{8}$ $\quad\Rightarrow\ \tan(\cos^{-1} f(6)) = \tan\frac{\pi}{8} = \sqrt{2} - 1$, satisfies $\alpha^2 + 2\alpha - 1 = 0$

$\therefore$ 1 is correct

$f(4)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$ 2 is correct

$f(5)=\cos\frac{\pi}{7} \Rightarrow \cos^{-1}f(5)=\frac{\pi}{7}$

$\therefore$ $\sin(7\cos^{-1}f(5))=\sin\pi=0$ 3 is correct

$\lim_{n\to\infty}\cos\left(\frac{\pi}{n+2}\right)=1$ 4 is incorrect

2. Let $P_1=I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $P_2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$, $P_3=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, $P_4=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$,

$P_5=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$, $P_6=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ and $X=\sum_{k=1}^{6}P_k\begin{bmatrix}2&1&3\\1&0&2\\3&2&1\end{bmatrix}P_k^T$

Where P_k^T denotes the transpose of the matrix P_k. Then which of the following options is/are correct ?

(1) If $X\begin{bmatrix}1\\1\\1\end{bmatrix}=\alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$, then $\alpha=30$

(2) X is a symmetric matrix

(3) X – 30I is an invertible matrix

(4) The sum of diagonal entries of X is 18

Ans. (1, 2, 4)

Sol. $P_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$\therefore \quad \begin{matrix} P_1 = & P_1^T = & P_1^{-1} \\ P_2 = & P_2^T = & P_2^{-1} \\ .... & .... & ...... \\ P_6 = & P_6^T = & P_6^{-1} \end{matrix}$$

And $A^T = A$ where A = $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

Using formula $(A+B)^T = A^T + B^T$

$$X = P_1AP_1^T + P_2AP_2^T.....P_6AP_6^T$$

$$X^T = (P_1AP_1^T + P_2AP_2^T.....+P_6AP_6)^T$$

$$\therefore \quad X^T = P_1AP_1^T + P_2AP_2^T.....+P_6AP_6^T$$

$\therefore$ X is symmetric

Let B = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$XB = P_1AP_1^TB + P_2AP_2^TB.....+P_6AP_6^TB$$

$$XB = P_1AB + P_2AB.....+P_6AB$$

$$XB = (P_1 + P_2.....+P_6)\begin{pmatrix} 6 \\ 3 \\ 6 \end{pmatrix}$$

$$P_1 + P_2 + P_3.....+P_6 = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$\therefore \quad XB = \begin{pmatrix} 2\times 6 & +2\times 3 & +2\times 6 \\ 2\times 6 & +2\times 3 & +2\times 6 \\ 2\times 6 & +2\times 3 & +2\times 6 \end{pmatrix} = \begin{pmatrix} 30 \\ 30 \\ 30 \end{pmatrix}$$

$$\therefore \quad XB = 30B \Rightarrow \alpha = 30$$

$\therefore \quad X\begin{pmatrix}1\\1\\1\end{pmatrix} = 30\begin{pmatrix}1\\1\\1\end{pmatrix}$ Choice (i) is correct

$\Rightarrow$ (X – 30 I) B = 0 has a non trivial solution $B = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\Rightarrow \quad |X - 30\,I| = 0 \qquad \therefore$ non-invertible

Trace (X) = $Tr(P_1AP_1^T) + Tr(P_2AP_2^T) .... + Tr(P_6AP_6^T)$

= $3+3+3+3+3+3$

= 18 $\qquad \therefore$ Ans. (4)

3. Let $f: R \to R$ be a function we say that f has

PROPERTY 1 if $\lim_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and

PROPERTY 2 if $\lim_{h\to 0}\frac{f(h)-f(0)}{h^2}$ exists and is finite.

The which of the following options is/are correct ?

(1) f(x) = sin x has PROPERTY 2
(2) f(x) = x |x| has PROPERTY 2
(3) f(x) = |x| has PROPERTY 1
(4) $f(x) = x^{2/3}$ has PROPERTY 1

Ans. (3, 4)

Sol. **(1)** $f(x) = \sin x$ with property - 2

$\lim_{h\to 0}\frac{\sin h - 0}{h^2} =$ does not exist.

So (1) can't be the answer.

(2) $f(x) = x|x|$ with property - 2

$\left.\begin{array}{l}\lim_{h\to 0^+}\frac{h^2-0}{h^2} = 1\\ \lim_{h\to 0^-}\frac{-h^2-0}{h^2} = -1\end{array}\right\}$ limit does not exists.

So (2) can't be the answer.

(3) $f(x)=|x|$ with property - 1

$$\left.\begin{aligned}&\lim_{h\to 0^+}\frac{|h|-0}{\sqrt{h}}=\sqrt{h}=0\\&\lim_{h\to 0^-}\frac{+|h|-0}{\sqrt{-h}}=+\sqrt{-h}=0\end{aligned}\right\}\text{so it holds.}$$

(4) $f(x)=x^{2/3}$ with property – 1

$$\left.\begin{aligned}&\lim_{h\to 0^+}\frac{h^{2/3}-0}{\sqrt{h}}=h^{\frac{2}{3}-\frac{1}{2}}=0\\&\lim_{h\to 0^-}\frac{h^{2/3}-0}{\sqrt{-h}}==0\end{aligned}\right\}\text{exists and finite.}$$

So (3) and (4) are the correct.

4. Three lines

$L_1:\quad \vec{r}=\lambda\hat{i},\lambda\in\mathbb{R},$

$L_2:\quad \vec{r}=\hat{k}+\mu\hat{j},\mu\in\mathbb{R}$ and

$L_3:\quad \vec{r}=\hat{i}+\hat{j}+v\hat{k},v\in\mathbb{R}$

are given. For which point(s) Q on L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear?

(1) $\hat{k}-\frac{1}{2}\hat{j}$

(2) $\hat{k}+\hat{j}$

(3) $\hat{k}+\frac{1}{2}\hat{j}$

(4) $\hat{k}$

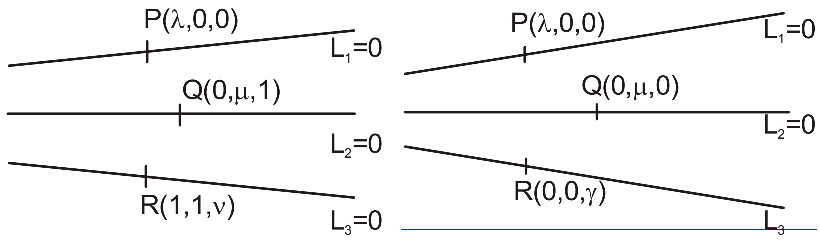
Ans. (1, 3)

Sol. $L_1=\vec{r}=\lambda\hat{i}\ :\ \lambda\in IR$

$L_2=\vec{r}=\vec{k}+\mu\hat{j}\ :\ \mu\in IR$

$L_3=\vec{r}=\hat{i}+\hat{j}+\nu\hat{k}\ :\ \nu\in IR$

If P, Q, R are collinear then



$\overrightarrow{PQ} = x \cdot \overrightarrow{PR}$, for some scalar 'x'

$\overrightarrow{PQ} = -\lambda\hat{i} + \mu\hat{j} + \hat{k}$ and $\overrightarrow{PR} = (1-\lambda)\hat{i} + \hat{j} + \nu\hat{k}$

as well as $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

So, $\frac{-\lambda}{1-\lambda} = \frac{\mu}{1} = \frac{1}{\nu}$ so $\nu \neq 1$

again $\mu \neq 0, 1$

So,the possible points which can not be the point Q are (0, 0, 1)

i.e. $\hat{k}$ and (0, 1, 1) i.e. $\left(\hat{j}+\hat{k}\right)$.

Hence possible values of Q are

$\left(\hat{k} - \frac{1}{2}\hat{j}\right)$ or $\left(\hat{k} + \frac{1}{2}\hat{j}\right)$

So. (1) and (3) are the correct answer.

5. Let $x \in \mathbb{R}$ and let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$

Then which of the following options is/are correct?

(1) det R = det $\begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix}$ + 8, for all $x \in \mathbb{R}$

(2) For x = 1, there exists a unit vector $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ for which R $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(3) There exists a real number x such that PQ = QP

(4) For x = 0, if R $\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6 \begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then a + b = 5

Ans. (1, 4)

Sol. $\det R = \det(P)\cdot\det(Q)\cdot\det(P^{-1})$

$= \det(Q)$

$= 48 - 4x^2$

Options (i) $\det(R) = 2(20-0) + x(0-4x) + 8$

$= 48 - 4x^2$ $\rightarrow$ Correct

Options (ii) $\Rightarrow\ \alpha = \beta = \gamma = 0$ $\rightarrow$ Wrong

Options (iii) $PQ = \begin{bmatrix} 2+x & 2x+4 & x+6 \\ 2x & 8+2x & 12 \\ 3x & 3x & 18 \end{bmatrix}$

$QP = \begin{bmatrix} 2 & 2+2x & - \\ - & - & - \\ - & - & - \end{bmatrix}$ $\Rightarrow 2+2x = 2x+4 \Rightarrow 2 = 4$ wrong

Option (iv) for $x = 0$

$R = PQP^{-1} \Rightarrow R = \begin{bmatrix} 2 & 1 & \frac{2}{3} \\ 0 & 4 & \frac{4}{3} \\ 0 & 0 & 6 \end{bmatrix}$

$\therefore\quad R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$

$2 + a + \frac{2}{3}b = 6$ and $4a + \frac{4b}{3} = 6a$

$\Rightarrow a = 2,\ b = 3$ $\rightarrow$ Correct.

6. Let $f(x)=\frac{\sin \pi x}{x^2},\ x>0$.

Let $x_1<x_2<x_3<\cdots<x_n<\cdots$ be all the points of local maximum of f

And $y_1<y_2<y_3<\cdots<y_n<\cdots$ be all the points of local minimum of f.

Then which of the following options is/are correct ?

(1) $x_{n+1}-x_n>2$ for every n

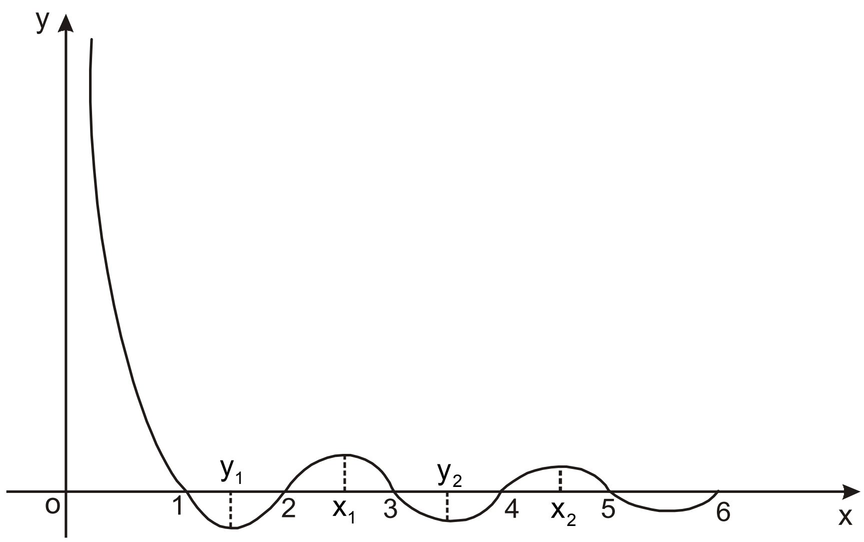
(2) $x_n\in\left(2n,\ 2n+\frac{1}{2}\right)$ for every n

(3) $|x_n-y_n|>1$ for every n

(4) $x_1<y_1$

Ans. (1, 2, 3)

Sol. The graph of f(x) is

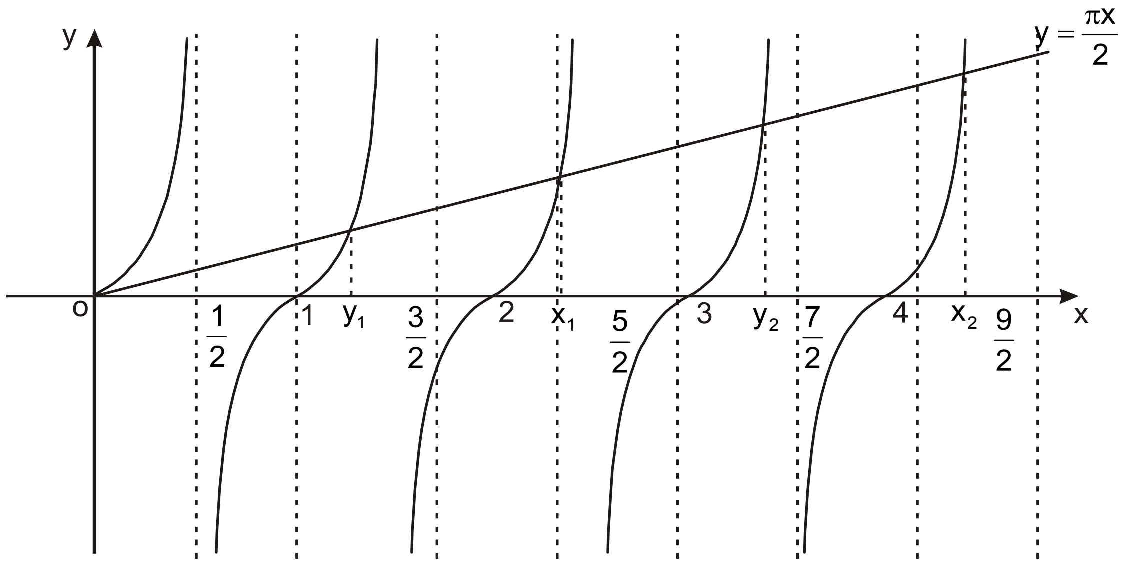


$$f'(x)=\frac{x^2\pi\cos\pi x-2x\sin\pi x}{x^4}$$

$$=\frac{x\pi\cos\pi x-2\sin\pi x}{x^3}$$

Now, $x\pi\cos\pi x-2\sin\pi x=0$

$$\Rightarrow\quad \tan\pi x=\frac{\pi x}{2}$$



Comparing the graphs of $y = \tan \pi x$ and $y = \frac{\pi x}{2}$

From figure,

$\Rightarrow \; x_{n+1} - x_n > 2 \forall n$

Similarly $\quad x_n \in \left(2n, 2n + \frac{1}{2}\right), \forall n$

Again $|x_1 - y_1| > 1$

From figure options (iv) is wrong

7. For $a \in \mathbb{R}, |a| > 1$, let $\lim_{n\to\infty} \left(\frac{1 + \sqrt[3]{2} + \cdots + \sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2} + \frac{1}{(an+2)^2} + \cdots + \frac{1}{(an+n)^2} \right)} \right) = 54.$

Then the possible value(s) of a is/are

(1) 8

(2) - 9

(3) - 6

(4) 7

Ans. (1, 2)

Sol. $$\lim_{n\to\infty}\left(\frac{1^{\frac{1}{3}}+2^{\frac{1}{3}}+........n^{\frac{1}{3}}}{n^{\frac{7}{3}}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+...\frac{1}{(an+n)^2}\right)}\right)=54$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{\left(\frac{1}{n}\right)^{\frac{1}{3}}+\left(\frac{2}{n}\right)^{\frac{1}{3}}+\left(\frac{n}{n}\right)^{\frac{1}{3}}}{\left(\frac{1}{\left(a+\frac{1}{n}\right)^2}+\frac{1}{\left(a+\frac{2}{n}\right)^2}+...+\frac{1}{\left(a+\frac{n}{n}\right)^2}\right)}\right)=54$$

$$\Rightarrow \lim_{x\to\infty}\frac{\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{\frac{1}{3}}}{\sum_{r=1}^{n}\frac{1}{\left(a+\frac{r}{n}\right)^2}}=54 \qquad \Rightarrow \frac{3a(a+1)}{4(a+1-a)}=54$$

$$\Rightarrow a^2+a=72 \Rightarrow a^2+a-72=0$$

$$\Rightarrow (a+9)(a-8)=0\text{ , } \boxed{a=-9, 8}$$

8. Let f; $\mathbb{R}\to\mathbb{R}$ be given by $f(x)=(x-1)(x-2)(x-5)$. Define $F(x)=\int_0^x f(t)dt,\ x>0$.

Then which of the following options is/are correct?

(1) F has a local maximum at x = 2

(2) F has two local maxima and one local minimum in $(0,\infty)$

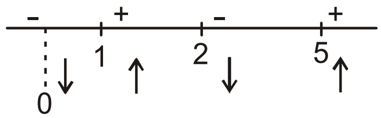
(3) F has a local minimum at x = 1

(4) $F(x)\neq 0$ for all $x\in(0,5)$

Ans. (1,3,4)

Sol. $F(x) = \int_0^x f(t)dt,\ x > 0$

$F'(x) = f(x) = (x-1)(x-2)(x-5)$

Sign scheme of $F'(x)$: 

F(x) has local minima at x = 1, 5

F(x) has local maxima at x = 2

Clearly, F(1) < 0

$F(2) = \int_0^2 f(t)at = \int_0^2 (t^3 - 8t^2 + 17t - 10)dt$

$= 4 - \frac{64}{3} + 34 - 20$

$= 38 - \frac{124}{3}$

F(2) is also $= \frac{114-124}{3} = \frac{-10}{3} < 0$

as after x = 2 upto x = 5

F(x) is decreasing.

$\therefore$ f(x) can never be zero for any $x \in (0,5)$.

$\therefore$ Correct options (1, 3, 4)

SECTION 2 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value of to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : **+3** If ONLY the correct numerical value is entered.

  Zero Marks **: 0** In all other cases.

1. The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals ______

Ans. (0.50)

Sol. $I = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ ......(i)

$I = \int_0^{\frac{\pi}{2}} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^5} d\theta$ ......(ii)

(i) + (ii)

$$2I = 3\int_0^{\frac{\pi}{2}} \frac{d\theta}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^4}$$

$$= 3\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\, d\theta}{\left(\sqrt{\tan\theta}+1\right)^4}$$

Let $\tan\theta = u^2$

$\sec^2\theta\, d\theta = 2udu$

$$\Rightarrow \quad 2I = 3\int_0^{\infty} \frac{2udu}{(u+1)^4} = 6\int_0^{\infty}\left(\frac{1}{(u+1)^3} - \frac{1}{(u+1)^4}\right)du$$

$$I = 3\left[-\frac{1}{2(u+1)^2} + \frac{1}{3(u+1)^3}\right]_0^{\infty}$$

$$= \left[0 - \left(-\frac{1}{2}+\frac{1}{3}\right)\right] = 3\times\frac{1}{6} = \frac{1}{2}$$

$\Rightarrow\ I = 0.50$

2. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ equals ___

Ans. (0)

Sol. $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$

$$= \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{1}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{\sin\left(\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)-\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\left(\tan\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)\right)$$

$$=\sec^{-1}\left(\frac{1}{4}\left(\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}\right)+\tan\left(\frac{7\pi}{12}+\frac{2\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)+\cdots+\tan\left(\frac{7\pi}{12}+\frac{11\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{10\pi}{2}\right)\right)\right)$$

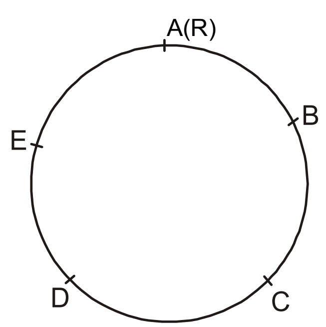
$$= \sec^{-1}\left(\frac{1}{4}\left(\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}\right)\right)\right)=\sec^{-1}\left(\frac{1}{4}\left(-\cot\frac{7\pi}{12}-\tan\frac{7\pi}{12}\right)\right)$$

$$= \sec^{-1}\left(\frac{-1}{4}\left(\frac{\cos\frac{7\pi}{12}}{\sin\frac{7\pi}{12}}+\frac{\sin\frac{7\pi}{12}}{\cos\frac{7\pi}{12}}\right)\right) = \sec^{-1}\left(\frac{-1}{4}\left(\frac{1}{\sin\frac{7\pi}{12}\cdot\cos\frac{7\pi}{12}}\right)\right) = \sec^{-1}\left(\frac{-1}{2}\frac{1}{\sin\frac{7\pi}{6}}\right) = 0$$

3. Five person A, B, C,D and E are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green, then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is ___

Ans. (30)

Sol.



Maximum number of hats used of same colour are 2.

When 1R, 2B, 2G

Required number of ways = ${}^5C_1 \times 2 = 10$

Other possibilities

1B, 2R, 2G

1G, 2R, 2B

Hence, total number of ways = $3 \times 5 \times 2 = 30$ ways.

4. Suppose $\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$

Holds for some positive integer n. then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals ___

Ans. (6.20)

Sol. $\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} \frac{n(n+1)}{2} & n(n+1) \cdot 2^{n-2} \\ n \cdot 2^{n-1} & 4^n \end{vmatrix} = 0$

$\Rightarrow \quad 2^{2n-1} n(n+1) - n^2(n+1)2^{2n-3} = 0$

$\Rightarrow \quad n(n+1)2^{2n-3}(4-n) = 0$

$\Rightarrow \quad n = 0, -1, 4$

Where n = 4 is valid

$\therefore \quad \sum_{k=0}^{4} \frac{{}^4C_k}{k+1} = \sum_{k=0}^{4} \frac{{}^5C_{k+1}}{5} = \frac{2^5 - 1}{5} = \frac{31}{5} = 6.20$

5. Let |X| denote the number of elements in a set X. Let S ={1,2,3,4,5,6} be a sample space, where each element is equally likely to occur. If A and B are independent events associated with S, then the number of ordered pairs (A, B) such that $1 \le |B| < |A|$, equals ___

Ans. (422)

Sol. $P\left(\frac{B}{A}\right) = P(B)$

$\Rightarrow \quad \frac{n(A \cap B)}{n(A)} = \frac{n(B)}{n(S)}$ ....(I)

$\Rightarrow$ n(A) should have 2 or 3 as prime factors.

$\Rightarrow$ n(A) can be 2, 3, 4 or 6 as $n(A) > 1$

n(A) = 2 does not satisfy the constraint (i)

for n(A) = 3, n(B) = 2 and $n(A \cap B) = 1$

$\Rightarrow$ No of ordered pair $= 6_{C_4} \times \frac{4!}{2!} = 180$

for n(A) = 4, n(B) = 3 and $n(A \cap B) = 2$

$\Rightarrow$ No of ordered pairs = $6_{C_5} \times \frac{5!}{2!\,2!} = 180$

for n(A) = 6, n(B) can be 1, 2, 3, 4, 5.

$\Rightarrow$ No of ordered pairs $= 2^6 - 2 = 62$

Total ordered pair = 180 + 180 + 62 = 422

6. Let $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$ and $\hat{b} = \hat{i} + 2\hat{j} + \hat{k}$ be two vectors. Consider a vector $\vec{c} = \alpha\vec{a} + \beta\vec{b}, \alpha, \beta \in \mathbb{R}$. If the projection of $\vec{c}$ on the vector $(\vec{a} + \vec{b})$ is $3\sqrt{2}$, then the minimum value of $(\vec{c} - (\vec{a} \times \vec{b})) \cdot \vec{c}$ equals____

Ans. (18)

Sol. $\vec{C} = (2\alpha + \beta)\hat{i} + \hat{j}(\alpha + 2\beta) + \hat{k}(\beta - \alpha)$

$\frac{\vec{c} \cdot (\vec{a} + \vec{b})}{|\vec{a} + \vec{b}|} = 3\sqrt{2}$

$\Rightarrow \quad \alpha + \beta = 2$ ....(i)

Now, $(\vec{c}-(\vec{a}\times\vec{b}))\cdot(\alpha\vec{a}+\beta\vec{b})=|\vec{c}|^2$

$|\vec{c}|^2=\alpha^2|\vec{a}|^2+\beta^2|b|^2+2\alpha\beta(\vec{a}\cdot\vec{b})$

$= \quad 6(\alpha^2+\beta^2+\alpha\beta)$

$= \quad 6(\alpha^2+(2-\alpha)^2+\alpha(2-\alpha))$

$= \quad 6((\alpha-1)^2+3)$

$\Rightarrow$ Min value = 18

SECTION 3 (Maximum Marks : 12)

- This section contains **TWO (02)** List-Match sets.
- Each List-Match set has **TWO (02)** Multiple Choice Questions**.**
- Each List-Match set has two lists : **LIST-I** and **LIST-II.**
- **LIST-I** has **Four** entries (I), (II), (III) and (IV) and **LIST-II** has **Six** entries (P), (Q), (R), (S), (T) and (U).
- **FOUR** options are given in each Multiple Choice Question based on **LIST-I** and **LIST-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : **+3** If ONLY the option corresponding to the correct combination is chosen.

  Zero Marks **: 0** If none of the options is chosen (i.e. the question is unanswered).

  Negative Marks : **–1** In all other cases.

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x)=\sin(\pi\cos x)$ and $g(x)=\cos(2\pi\sin x)$ be two functions defined for x > 0. Define the following sets whose elements are written in the increasing order :

$$X=\{x:f(x)=0\},\qquad Y=\{x:f'(x)=0\}$$

$$Z=\{x:g(x)=0\},\qquad W=\{x:g'(x)=0\}.$$

List-I contains the sets X,Y,Z and W. List-II contains some information regarding these sets.

List-I		**List-II**	
(I)	X	**(P)**	$\supseteq\left\{\frac{\pi}{2},\frac{3\pi}{2},4\pi,7\pi\right\}$
(II)	Y	**(Q)**	an arithmetic progression
(III)	Z	**(R)**	NOT an arithmetic progression
(IV)	W	**(S)**	$\supseteq\left\{\frac{\pi}{6},\frac{7\pi}{2},\frac{13\pi}{6}\right\}$
		(T)	$\supseteq\left\{\frac{\pi}{3},\frac{2\pi}{3},\pi\right\}$
		(U)	$\supseteq\left\{\frac{\pi}{6},\frac{3\pi}{4}\right\}$

1. Which of the following is the only CORRECT combination ?

(1) (IV), (Q), (T)

(2) (IV), (P), (R), (S)

(3) (III), (P), (Q), (U)

(4) (III), (R), (U)

Ans. (2)

2. Which of the following is the only CORRECT combination ?

(1) (II), (Q), (T)

(2) (I), (Q), (U)

(3) (II), (R), (S)

(4) (I), (P), (R)

Ans. (1)

Sol. 1.- 2. $f(x)=\sin(\pi\cos x), g(x)=\cos(2\pi\sin x)$ for x > 0

$X = \{x : f(x)=0\} \quad \Rightarrow \quad \sin(\pi\cos x)=0$

$\Rightarrow \quad \pi\cos x = n\pi$

$\Rightarrow \quad \cos x = n$

$\Rightarrow \quad \cos x = 0,-1,1$

$\Rightarrow \quad x=\frac{k\pi}{2}, k\in I$

But x > 0 $\Rightarrow$ $\boxed{X=\left\{\frac{k\pi}{2},k\in N\right\}}$

Y = $\{x:f'(x)=0\}$ $\Rightarrow$ $\cos(\pi\cos x)(-\pi\sin x)=0$

$\Rightarrow$ $\cos(\pi\cos x)=0$ or $\sin x=0$

$\Rightarrow$ $\pi\cos x=(2n+1)\frac{\pi}{2}$ or $x=n\pi$

$\Rightarrow$ $\cos x=\frac{(2n+1)}{2}$ or $x=n\pi$

$\Rightarrow$ $\cos x=\pm\frac{1}{2}$

$\Rightarrow$ $x=k\pi\pm\frac{\pi}{3}$ or $x=n\pi$

$\Rightarrow$ $\boxed{Y=\left\{x:n\pi,k\pi\pm\frac{\pi}{3},\ n,k\in I\ \&\ x>0\right\}.}$

Z = $\{x:g(x)=0\}$ $\Rightarrow$ $\cos(2\pi\sin x)=0$

$\Rightarrow$ $(2\pi\sin x)=(2n+1)\frac{\pi}{2}$

$\Rightarrow$ $\sin x=\frac{(2n+1)}{4}$

$\Rightarrow$ $\sin x=\pm\frac{1}{4},\pm\frac{3}{4}$

$\Rightarrow$ $x=k\pi\pm\sin^{-1}\frac{1}{4},\ k\pi\pm\sin^{-1}\frac{3}{4}$

$\boxed{Z=\left\{k\pi\pm\sin^{-1}\frac{1}{4},k\pi+\sin^{-1}\frac{3}{4},k\in I\ \&\ x>0\right\}}$

W = $\{x:g'(x)=0\}$ $\Rightarrow$ $-\sin(2\pi\sin x)2\pi\cos x=0$

$\Rightarrow$ $\sin(2\pi\sin x)=0$ or $\cos x=0$

$$\Rightarrow \quad 2\pi \sin x = n\pi \text{ or } x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad \sin x = \frac{n}{2} \text{ or } x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad \sin x = 0, -1, 1, \pm\frac{1}{2}$$

$$\Rightarrow \quad x = \frac{k\pi}{2}, k\pi \pm \frac{\pi}{6} \text{ or } x = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow \quad x = \frac{k\pi}{2}, n\pi \pm \frac{\pi}{6}$$

$$\Rightarrow \boxed{W = \left\{\frac{k\pi}{2}, n\pi \pm \frac{\pi}{6}, n, k \in I \;\&\; x > 0\right\}}$$

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circles $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3 and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expressions given in the List-I whose values are given in List-II below :

	List-I		**List-II**
(I)	$2h + k$	**(P)**	6
(II)	$\frac{\text{Length of ZW}}{\text{Length of XY}}$	**(Q)**	$\sqrt{6}$
(III)	$\frac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	**(R)**	$\frac{5}{4}$
(IV)	α	**(S)**	$\frac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\frac{10}{3}$

3. Which of the following is the only INCORRECT combination

(1) (I), (P)
(2) (IV), (S)
(3) (III), (R)
(4) (IV), (U)

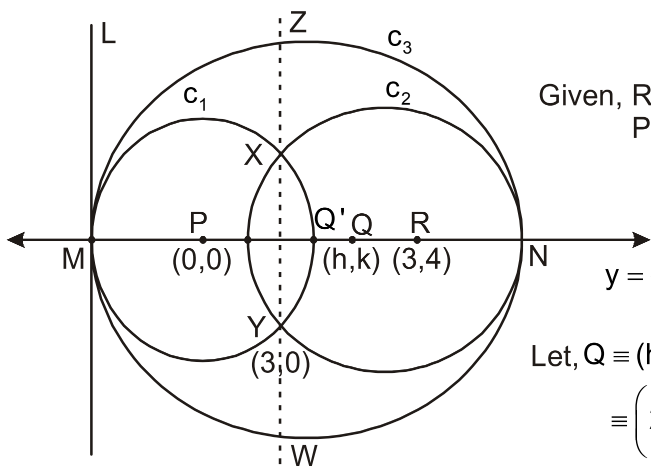
Ans. (2)

4. Which of the following is the only CORRECT combination

(1) (II), (T)
(2) (II), (Q)
(3) (I), (S)
(4) (I), (U)

Ans. (2)

Sol. (3- 4)



$y = \frac{4}{3}x$

Given, RN = 4
PM = 3

$\text{Let, } Q \equiv (h,k)$

$\equiv \left(\lambda, \frac{4}{3}\lambda\right)$

and QN = QM = r

Now, QN = QR + RN

$$r = \sqrt{(3-\lambda)^2 + \left(4 - \frac{4}{3}\lambda\right)^2} + 4 \qquad \ldots\ldots(i)$$

and QM = QP + PM

$$r = \sqrt{\lambda^2 + \left(\frac{4}{3}\lambda\right)^2} + 3 \qquad \ldots\ldots(ii)$$

from (i) and (ii)

$$\sqrt{(3-\lambda)^2+\left(4-\frac{4}{3}\lambda\right)^2}+4=\sqrt{\lambda^2+\left(\frac{4}{3}\lambda\right)^2}+3$$

$$\Rightarrow \quad \boxed{\lambda=\frac{9}{5}}$$

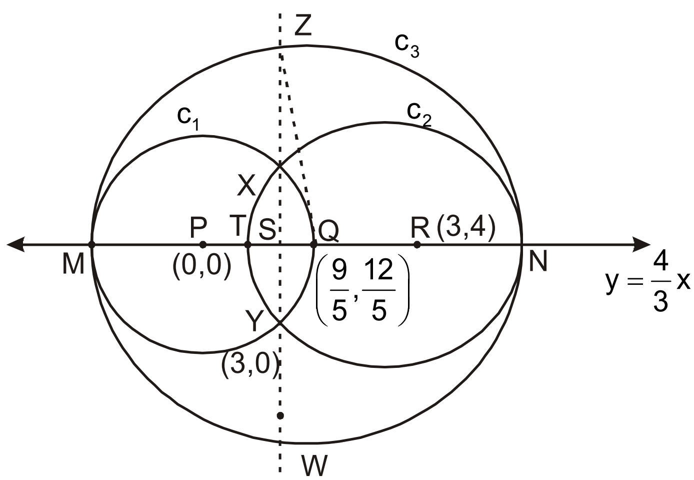
Hence, $Q\equiv(h,k)\equiv\left(\frac{9}{5},\frac{12}{5}\right)\Rightarrow\boxed{2h+k=6}$

$\boxed{\text{I},(\text{P})}$

r = 6

$$C_3\equiv\left(x-\frac{9}{5}\right)^2+\left(y-\frac{12}{5}\right)^2=36$$

As, $MQ'=6$ and MQ = 6 $\Rightarrow$ Q concides with Q'



Eqn. of line through the point X and Y is $3x+4y=9$

$$\Rightarrow \quad S=\left(\frac{27}{25},\frac{36}{25}\right)$$

$$SQ=\sqrt{\left(\frac{27}{25}-\frac{9}{5}\right)^2+\left(\frac{36}{25}-\frac{12}{5}\right)^2}=\frac{6}{5}$$

$$ZS=\sqrt{(ZQ)^2-(SQ)^2}=\sqrt{36-\frac{36}{25}}=\frac{12\sqrt{6}}{5} \quad \Rightarrow \boxed{ZW=\frac{24\sqrt{6}}{5}}$$

& $\quad SP = \sqrt{\left(\frac{27}{25}\right)^2 + \left(\frac{36}{25}\right)^2} = \frac{9}{5}$

$SM = \frac{9}{5} + 3 = \frac{24}{5}$

$SY = \frac{|4 \times 3 - 3 \times 0|}{\sqrt{3^2 + 4^2}} = \frac{12}{5}$

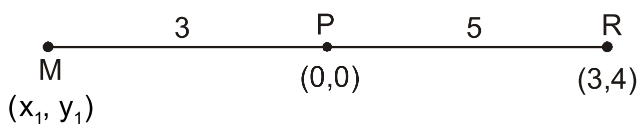
$\Rightarrow \quad \boxed{XY = \frac{24}{5}}$

$\Rightarrow \quad \frac{\text{length of } ZW}{\text{length of } XY} = \frac{\left(\frac{24\sqrt{6}}{5}\right)}{\left(\frac{24}{5}\right)} = \sqrt{6}$

(II, Q)

Also, $\boxed{\frac{\text{area of } \Delta MZN}{\text{area of } \Delta ZMW} = \frac{\frac{1}{2} \times SZ \times MN}{\frac{1}{2} \times SM \times ZW} = \frac{12}{2 \times \frac{24}{5}} = \frac{5}{4}}$

$\boxed{\text{III, R}}$



$5x_1 + 3 \cdot 3 = 0 \qquad \Rightarrow x_1 = -\frac{9}{5}$

$5y_1 + 3 \cdot 4 = 0 \qquad \Rightarrow y_1 = -\frac{12}{5}$

Eqn. of line L or common tangent to C_1 and C_3 is,

$y + \frac{12}{5} = -\frac{3}{4}\left(x + \frac{9}{5}\right)$

$\Rightarrow \quad \boxed{3x + 4y + 15 = 0}$

Now, the line L is also tangent to the parabola $x^2 = 8\alpha y \Rightarrow x^2 = 8\alpha\left(\frac{15 + 3x}{-4}\right)$

$x^2 + 6\alpha x + 30\alpha = 0 \Rightarrow 36\alpha^2 - 4.30\alpha = 0$

$\alpha(3\alpha - 10) = 0 \Rightarrow \alpha = 0$ or $\boxed{\alpha = \frac{10}{3}}$

(IV, U)