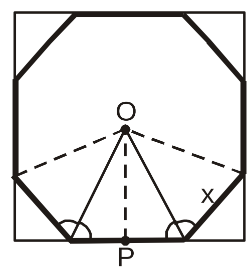
PRMO_2019

1. From a square with sides of length 5, triangular pieces from the four corners are removed to from a regular octagon. Find the area removed to the nearest integer ?

Ans. 04

Sol.



The angle subtended at the center by a side is $\frac{\pi}{4}$, $OP = \frac{5}{2}$, therefore, side of the regular octagon $x = 2 \cdot \frac{5}{2} \tan \frac{\pi}{8}$

$$= 5(\sqrt{2} - 1)$$

$\therefore$ removed area $= 4 \cdot \frac{1}{2} \cdot \frac{x}{\sqrt{2}} \cdot \frac{x}{\sqrt{2}}$

$$= x^2$$

$$= 25(3 - 2\sqrt{2})$$

2. Let $f(x) = x^2 + ax + b$. If for all nonzero real x $f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and the roots of f(x) = 0 are integers, what is the value of $a^2 + b^2$?

Ans. 13

Sol. $f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \; \forall \; x \in R - \{0\}$

(I)

Put f(x) $= x^2 + ax + b$ in (I) to get b = 2

Now $x^2 + ax + 2 = 0$ has both roots integers with product of roots = 2, therefore, $a = \pm 3$

hence $a^2 + b^2 = 13$.

3. Let x_1 be a positive real number and for every intger $n \geq 1$ let $x_{n+1} = 1 + x_1 x_2 \ldots x_{n-1} x_n$. If $x_5 = 43$, what is the sum of digits of the largest prime factors of x_6?

Ans. 13

Sol. $x_{x+1} = 1 + x_1x_2x_3x_4...x_x \quad \forall n \geq 1$

$x_5 = 1 + x_1x_2x_3x_4$

$\because x_5 = 43$

$\therefore x_1x_2x_3x_4 = 42$

Now, $x_6 = 1 + x_1x_2x_3x_4x_5$

$= 1 + 42 \cdot 43$

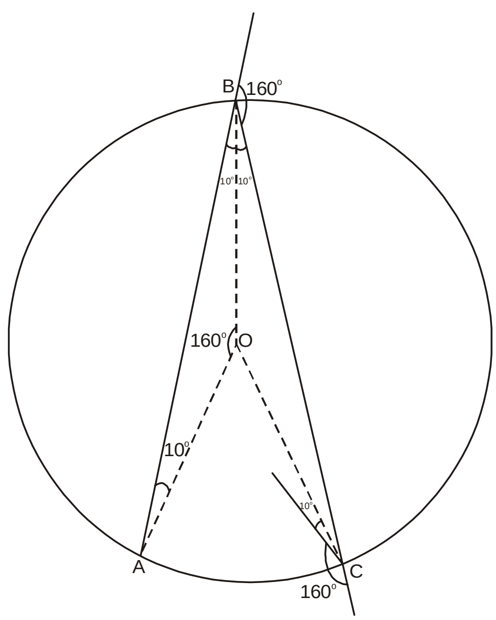
$= 1807$

$= 13 \times 139$

4. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160^o turn to the right and walks 4 more feet. It then makes another 160^o turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked ?

Ans. 36

Sol.



Let the ant starts at A, moves 4 feet to reach B, then makes 160^o turn and moves 4 feet to reach C.

Draw the circumcircle of ΔABC with centre at O $\angle AOB = 20^o$

$\therefore \angle OAB = \angle ABO = \angle OBC = 10^o \; (\because AB = BC)$

$\therefore \angle AOB = 180^o - 20^o = 160^o$

So after each move of 4 feat the radius joining position of ant to centre O rotates 160^o anticlockwise.

$\therefore$ ant reaches anthill again in m moves of 4 feat iff

$160^o \times m = 360^o \times n$

$\Leftrightarrow$ 4m = 9n

$\Leftrightarrow \frac{m}{9} = \frac{n}{4} = k$ (say)

$\therefore m = 9k, n = 4k$

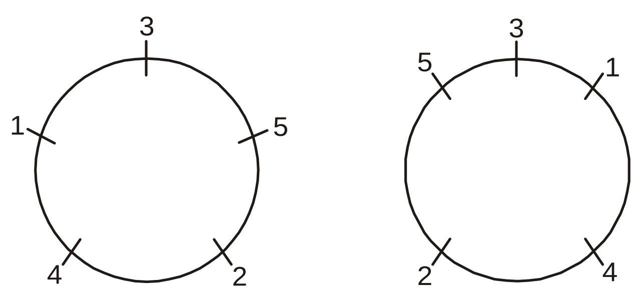
$\therefore$ least value of m = 9.

so distance covered = $9 \times 4 = 36$ feet.

5. Five persons wearing badges with numbers 1, 2, 3, 4, 5 are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other ? (Two arrangements obtained by rotation around the table are considered different.)

Ans. 10

Sol. Neighbours of person with badge number 3 can be 1 and 5 only. After placing persons with badge numbers 3, 1 and 5, the persons with badge numbers 2 and 4 can be placed in only one way.



Thus person with badge number 3 can be seated in 5 ways after which remaining can be seated in only 2 ways.

So total no of ways $= 5 \times 2 = 10$

6. Let $\overline{abc}$ be a three digit number with nonzero digits that $a^2 + b^2 = c^2$. What is the largest possible prime factor of abc ?

Ans. 29

Sol. The three digit number abc can only be 345 or 435

Now,

$345 = 3 \times 5 \times 23$

and $435 = 3 \times 5 \times 29$

7. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference in minutes between these two instants is written as $a+\frac{b}{c}$, where a, b, c are positive integers, with b < c and b/c in the reduced form. What is the value of a + b + c ?

Ans. 51

Sol. The hour hand moves 0.5^o in one minute and the minute hand moves 6^o in one minute, therefore, to meet the required condition after x minute we have

$6x=\frac{x}{2}+90$ or $6x=\frac{x}{2}+270$

i.e. $x=\frac{180}{11}$ or $x=\frac{540}{11}$

hence the required difference $=\frac{540}{11}-\frac{180}{11}=\frac{360}{11}=32+\frac{8}{11}$

8. How many positive integers n are there such that $3\le n\le 100$ and $x^{2^n}+x+1$ is divisible by x^2+x+1?

Ans. 49

Sol. Let $f(x)=x^{2^n}+x+1$, $3\le n\le 100$

If n is odd say $n=2p+1$

then

$f(x)=x^{2^{2p+1}}+x+1$

$=x^{3m+2}+x+1$

$\therefore f(\omega)=0$ and $f(w^2)=0$ where ω and ω^2 are complex cube roots of unity.

Hence f(x) is divisible by $(x-\omega)(x-\omega^2)$ i.e x^2+x+1

If n is even say n = 2p then

$f(x)=x^{2^{2p}}+x+1$

$=x^{3\lambda+1}+x+1$

$\therefore$neither $f(\omega)\neq 0$ nor $f(\omega^2)\neq 0$

Hence f(x) is not divisible by x^2+x+1

$\therefore$ Required values of n are the odd integers such that $3\le n\le 100$

9. Let the rational number p/q be closest to but not equal to 22/7 among all rational numbers with denominator < 100. What is the value of p – 3q ?

Ans. 14

Sol. $\frac{p}{q}-\frac{22}{7}=\frac{7p-22q}{7q}$

Let $|7p-22q|=1$

$7p=22q\pm 1=21q+(q\pm 1)$

$\therefore 7 \text{ divides} (q\pm 1)$

largest q < 100 can be 99,

for which $p=\frac{22\times 99-1}{7}=311$

$\therefore \frac{p}{q}=\frac{311}{99}$

so that $\left|\frac{7\times 311-22\times 99}{7\times 99}\right|=\frac{1}{7\times 99}$

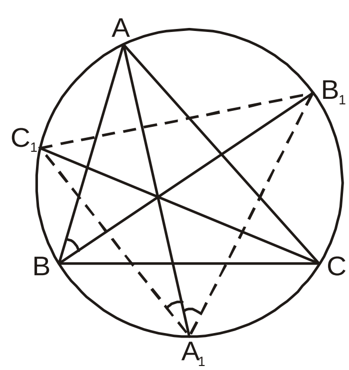
clearly $\frac{311}{99}$ is close if to $\frac{22}{7}$ for q < 100.

$\therefore p-3q=311-297=14.$

10. Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intercsect Ω at $A_1, B_1,$ and $C_1,$ respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1 B_1 C_1$ intersect Ω at A_2, B_2 and $C_2,$ respectively. If the smallest angle of triangle ABC is $40^o,$ what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees ?

Ans. 55

Sol.



$A_1=\angle B_1A_1A+\angle AA_1C_1$

$\quad=\angle B_1BA+\angle ACC_1$

$=\frac{B}{2}+\frac{C}{2}$

$=90^o-\frac{A}{2}$

The angles of the $\Delta A_1B_1C_1$ will be $90^o-\frac{A}{2}, 90^o-\frac{B}{2}, 90-\frac{C}{2}$ Similarly angles of $\Delta A_2\ B_2\ C_2$ will be $90^o-\frac{A_1}{2}, 90^o-\frac{B_1}{2}, 90^o-\frac{C_1}{2}$ Let A be 40^o, the smallest angle of ΔABC, hence $A_1=90^o-\frac{A}{2}=70^o$ will be largest angle of $\Delta A_1B_1C_1$ & therefore, smallest angle of $\Delta A_2B_2C_2$ is $A_2=90^o-\frac{A_1}{2}$

11. How many distince triangle ABC are there, up to similarity, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy $\cos A \cos B+\sin A \sin B \sin kC=1$ for some positive integer k, where kC does not exceed 360^o ?

11. (6)

$\text{Cos}A\,\text{Cos}B+\text{Sin}A\,\sin B\,\text{Sin}kC=1$

$\text{Sin}kC=\frac{1-\text{Cos}A\,\text{Cos}B}{\text{Sin}A\,\text{Sin}B}\le 1$

$\Rightarrow\ \text{Cos}(A-B)\ge 1$

$\Rightarrow\ \text{Cos}(A-B)=1$

$\Rightarrow$ A = B

$\therefore$ Sin kC = 1

$\Rightarrow$ kC = 90º

$\therefore$ Number of triangles is same as no of Even +ve integral divisors of 90 i.e. 6.

12. A natural number k > 1 is called good if there exist natural numbers $a_1<a_2<\cdots<a_k$ such that $\frac{1}{\sqrt{a_1}}+\frac{1}{\sqrt{a_2}}+...+\frac{1}{\sqrt{a_k}}=1.$ Let f(n) be the sum of the first n good numbers, $n\ge 1.$ Find the sum of all values of n for which $f(n+5)/f(n)$ is integer.

12. (18)

Let $\frac{1}{\sqrt{a_1}}+\frac{1}{\sqrt{a_2}}=1$, then $\frac{1}{\sqrt{a_2}}=1-\frac{1}{\sqrt{a_1}}=\frac{\sqrt{a_1}-1}{\sqrt{a_1}}$

$\Rightarrow \sqrt{a_2} = \frac{\sqrt{a_1}}{\sqrt{a_1} - 1}$

$\Rightarrow a_2 = \frac{a_1}{a_1 + 1 - 2\sqrt{a_1}} \notin I$ when $\sqrt{a_1} \notin I$.

Also, $1 = \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} < \frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_1}} = \frac{2}{\sqrt{a_1}}$

$\Rightarrow a_1 < 4$

$\Rightarrow a_1 = 2$ or 3

So, 2 is not a good number.

Now, $\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{36}} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$

$\therefore$ 3 is a good number.

$\because \frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

$\therefore \frac{1}{2k} = \frac{1}{3k} + \frac{1}{6k}$

$\therefore \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18}$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{54}$

and so on. In general $\frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^{k-2}} + \frac{1}{2.3^{k-2}} = 1 \ \forall k \geq 3$.

So, each k > 3 is a good number.

$$\therefore f(n) = \begin{cases} 0, & n = 1, 2 \\ 3 + 4 + .. + n, & n \geq 3 \end{cases}$$

$$\frac{f(n+5)}{f(n)} = \frac{3 + 4 + \ldots + (n+5)}{3 + 4 + \ldots + n} \ (\because f(n) \neq 0)$$

$$= \frac{\frac{(n+5)(n+6)}{2} - 3}{\frac{n(n+1)}{2} - 3} = \frac{n^2 + 11n + 24}{n^2 + n - 6} = \frac{(n+3)(n+8)}{(n+3)(n-2)}$$

$$= 1 + \frac{10}{n-2}$$

Which is integer for n – 2 = 1, 2, 5, 10

Sum of possible values of n = 18.

13. Each of the numbers $x_1, x_2, ..., x_{101}$ is ± 1. What is the smallest positive value of $\sum_{1 \le i < j \le 101} x_i x_j$?

13. (10)

$$\sum_{1 \le i < j \le 101} x_i x_j = \frac{(x_1 + x_2 + ... + x_{101})^2 - (x_1^2 + x_2^2 + ... + x_{101}^2)}{2} = \frac{\left(\sum x_1\right)^2 - 101}{2}$$

$\therefore$ Smallest positive value of $\sum_{1 \le i < j \le 101} x_i x_j = 10$.

14. Find the smallest positive integer $n \ge 10$ such that n + 6 is prime and 9n + 7 is perfect square.

14. (53)

Let $9n + 7 = m^2 \quad m \in I^+$

$\therefore$ $m^2 = 7 \pmod 9$

$\Rightarrow$ m = 9 P $\pm$ 4

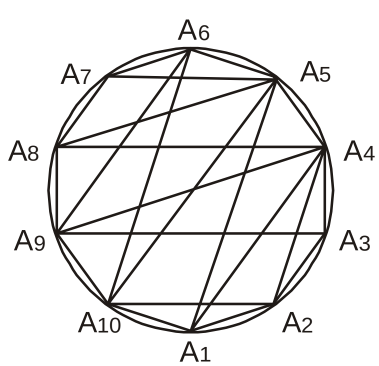
If P = 1 then m = 13 or 5 $\Rightarrow$ n = 18 or 2 but n does not satisfy the required criteria.

If P = 2 then m = 22 or 14 $\Rightarrow$ n = 53 or 21

$\therefore$ required value of n = 53.

15. In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected ?

15. (45)



The number of ways is $5.{}^4C_2 + 5.{}^3C_2 = 30 + 15 = 45$.

16. A pen costs Rs. 13 and a note book costs Rs. 17. A school spends exactly Rs. 10000 in the year 2017-18 to buy x pens and y note books such that x and y are as close as possible (i.e., |x – y| is minimum). Next year, in 2018-19, the school spends a little more than Rs. 10000 and buys y pens and x note books. How much **more** did the school pay ?

16. (40)

Here $13x + 17y = 10000$

$\Rightarrow 13x = 10000 - 17y$

$= 13 \times 769 - 13y + 3 - 4y$

$\therefore$ 13/4y – 3

Let $4y - 3 = 13k$

$$\Rightarrow y = \frac{13k+3}{4} = 3k + 1 + \frac{k-1}{4}$$

Let $k - 1 = 4m$

$\therefore y = 3(4m+1) + 1 + m$

$= 13m + 4$

$$\text{and } x = \frac{10000 - 17(13m+4)}{13}$$

$= 764 - 17m$

$\therefore |x - y| = |764 - 17m - 13m - 4| = |760 - 30m|$

which is minimum for m = 25 and So x = 339, y = 329

$\therefore$ In 2018 – 19, extra expenditure

$= 13y + 17x - 10000$

$= 13 \times 329 + 17 \times 339 - 10000$

$= 40.$

17. Find the number of ordered triples (a, b, c) of positive integers such that $30a + 50b + 70c \le 343$.

17. **(30)**

$30a + 50b + 70c \le 343$, a, b, c $\in I^+$

$30 + 50 + 70c \le 343$

$$\Rightarrow c \le \frac{263}{70}$$

$\therefore$ c = 1, 2 or 3

If c = 1, then $30a + 50b \le 273$

b = 1, a = 1, 2, ..., 7

b = 2, a = 1, 2, ..., 5

b = 3, a = 1, 2, 3, 4

b = 4, a = 1, 2

If c = 2, then $30a + 50b \le 203$

b = 1, a = 1, 2, ..., 5

b = 2, a = 1, 2, 3

b = 3, a = 1

If c = 3, then $30a + 50b = 133$

b = 1, a = 1, 2

b = 2, a = 1

$\therefore$ Total no. of triplets (a, b, c) = 7 + 5 + 4 + 2 + 5 + 3 + 1 + 2 + 1 = 30.

18. How many ordered pairs (a, b) of positive integers with a < b and $100 \leq a, b \leq 1000$ satisfy gcd(a, b) : lcm (a,b) = 1 : 495 ?

18. (44)

Let gcd (a, b) = g, then a = g.m and b = g.n, where m, n $\in N$ such that gcd (m, n) = 1.

$\therefore$ lcm (a, b) = g mn

$$\therefore \frac{\gcd(a,b)}{\text{lcm}(a,b)} = \frac{g}{gmn} = \frac{1}{495}$$

$\Rightarrow$ mn = 495 ... (1)

Also $100 \leq gm < gn \leq 1000$

Thus only possibilities are

(i) m = 9, n = 55, $12 \leq g \leq 18$

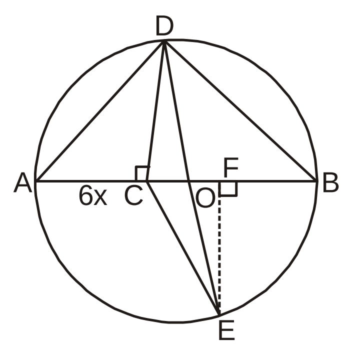
(ii) m = 11, n = 45, $10 \leq g \leq 22$

(iii) m = 15, n = 33, $7 \leq g \leq 30$

$\therefore$ no. of triplets (a, b, c) = 7 + 13 + 24 = 44.

19. Let AB be a diameter of a circle and let C be a point on the segment AB such that AC : CB = 6 : 7. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If [XYZ] denotes the area of the triangle XYZ, find [ABD]/[CDE] to the nearest integer.

19. (13)



Let O be the centre. Draw $EF \perp AB$.

Let AC = 6x and BC = 7x.

$$AO = OB = \frac{13x}{2}$$

$$CO = AO - OC = \frac{13x}{2} - 6x = \frac{x}{2}$$

Now $\Delta DCO \cong \Delta EFO$

$\therefore$ EF = DC

$$\therefore \frac{[ABD]}{[CDE]} = \frac{[ABD]}{[DCO]+[COE]} = \frac{\frac{1}{2}\times AB\times DC}{\frac{1}{2}\times CO\times DC+\frac{1}{2}\times CO\times FE} = \frac{\frac{1}{2}\times AB\times DC}{\frac{1}{2}\times CO\times DC+\frac{1}{2}\times CO\times DC}$$

$$= \frac{AB}{2\times CO} = \frac{13x}{2\times\frac{x}{2}} = 13.$$

20. Consider the set E of all natural numbers n such that when divided by 11, 12, 13, respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.

20. (Bonus)

The only possibilities for remainders are (3, 5, 7) or (3, 7, 11).

Let $N = 11k_1 + 3 = 12k_2 + 3 + d = 13k_3 + 3 + 2d$, where $k_1, k_2, k_3 \in I$ and d = 2 or 4.

$11k_1 + 3 = 12k_2 + 3 + d \Rightarrow 11k_1 = 12k_2 + d = 11k_2 + (k_2 + d)$

$\therefore$ $11/k_2 + d$

Again $12k_2 + 3 + d = 13k_3 + 3 + 2d$

$\Rightarrow$ $13k_3 = 12k_2 - d = 13k_2 - (k_2 + d)$

$\therefore$ $13/k_2+d$

$\therefore$ $143/k_2+d$

Let $k_2 = 143k - d,\ k \in I$

$\therefore$ $N = 1716k + 3 - 11d,\ k \in I^+$

So largest N does not exist.

There should be smallest N in the question which is 1675.

21. Consider the set E = {5, 6, 7, 8, 9}. For any partition {A, B} of E, with both A and B non-empty. consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N.

21. (17)

Let p be the sum of the product of elements of A and that of B. consider those partitions {A,B} for which p is prime. Let $6 \in A$, then $8, 9 \notin B$ as otherwise p will not be a prime.

so, $6, 8, 9 \in A$.

Case -I : $A = \{5, 6, 8, 9\},\ B = \{7\}$

p = 2167

but 11|p

Case -II : $A = \{6, 7, 8, 9\},\ B = \{5\}$

p = 3029 but 13|p

Case - III : $A = \{6, 8, 9\},\ B = \{5, 7\}$

p = 467, which is prime.

$\therefore$ $N = 467$

sum of digits of N = 17.

22. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599}\frac{1}{\sqrt{n}}$?

22. (78)

$$\frac{1}{\sqrt{n}}=\frac{2}{\sqrt{n}+\sqrt{n}}<\frac{2}{\sqrt{n}+\sqrt{n-1}}=2(\sqrt{n}-\sqrt{n-1})$$

again $\frac{1}{\sqrt{n}}=\frac{2}{\sqrt{n}+\sqrt{n}}>\frac{2}{\sqrt{n}+\sqrt{n+1}}=2(\sqrt{n+1}-\sqrt{n})$

$$\therefore\quad 2(\sqrt{n+1}-\sqrt{n})<\frac{1}{\sqrt{n}}<2(\sqrt{n}-\sqrt{n-1})$$

Putting $n=1,2,....,1599$ and adding,

$$2(\sqrt{1600}-\sqrt{2})<\sum_{n=2}^{1599}\frac{1}{\sqrt{n}}<2(\sqrt{1599}-\sqrt{1})$$

$$\Rightarrow\ 81-2\sqrt{2}<\sum_{n=1}^{1599}\frac{1}{\sqrt{n}}<2\sqrt{1599}-1$$

$$\Rightarrow\ 81-2\sqrt{2}<\sum_{n=1}^{1599}\frac{1}{\sqrt{n}}<2\sqrt{1599}-1$$

$$\Rightarrow\ 78<\sum_{n=1}^{1599}\frac{1}{\sqrt{n}}<79$$

$\therefore$ greatest integer less than $\sum_{n=1}^{1599}\frac{1}{\sqrt{n}}$ is 78.

23. Let ABCD be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of ABCD is the least. If $\{PA,PB,PC,PD\}=\{3,4,6,8\}$, what is the maximum possible area of ABCD ?

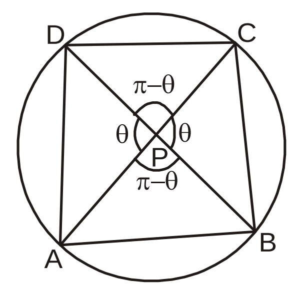
23. (55)

Using triangle inequality,

$PA+PC\geq AC$ and $PB+PD\geq BD$

$\therefore$ $PA+PB+PC+PD$ is least when

P is the point of intersection of diagonals of quadrilateral ABCD.



$\because$ ABCD is cyclic

$\therefore$ $PA.PC=PB.PD$

so we can take PA=3, PB=4, PC=8 and PD=6 let $\angle APD = \theta$, there

ar (quad. ABCD) = ar $(\Delta APB) + $ ar $(\Delta BPC) + $ ar$(\Delta CPD) + $ ar (ΔAPD)

$$= \frac{1}{2} \times 3 \times 4 \sin(\pi - \theta) + \frac{1}{2} \times 4 \times 8 \sin(\theta) + \frac{1}{2} \times 8 \times 6 \sin(\pi - \theta) + \frac{1}{2} \times 6 \times 3 \sin\theta$$

$$= \frac{1}{2}(12 + 32 + 48 + 18)\sin\theta$$

$$= 55\sin\theta$$

$\therefore$ area of quadrilateral ABCD is maximum when $\theta = 90^\circ$ and so maximum are =55 sq. units.

24. A 1 × n rectangle (n $\geq$ 1) is divided into n unit (1 × 1) squares. Each square of this rectangle is coloured red, blue or green. Let f(n) be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of f(9)/f(3) ? (The number of red squares can be zero.)

24. (37)

First we colour any 2K squares with red in ${}^nC_{2k}$ ways, then each of remaining n-2k squares can be coloured either blue or green in 2^{n-2k} ways. So total number of colouring when 2K squares are red is $n_{C_{2K}}(2^{n-2k})$

$$\therefore \quad f(n) = \sum_{k=0}^{[n/2]} {}^nC_{2k}(2^{n-2k})$$

$$= {}^nC_0 2^n + {}^nC_2 2^{n-2} + {}^nC_4 2^{n-4} + \ldots$$

$$= \frac{(2+1)^n + (2-1)^n}{2} = \frac{3^n + 1}{2}$$

$$\therefore \quad \frac{f(9)}{f(3)} = \frac{3^9 + 1}{3^3 + 1} = \frac{(3^3+1)(3^6 - 3^3 + 1)}{(3^3 + 1)} = 703$$

$\therefore$ Largest prince factor = 37.

25. A village has a circular wall around it, and the wall has four gates pointing north, south, east and west. A tree stands outside the village, 16 m north of the north gate, and it can be just seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter, in meters, of the wall that surrounds the village ?

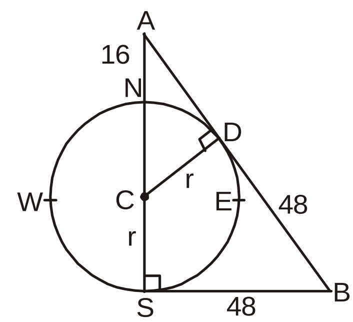
25. (48)

From ΔADC

$$AD = \sqrt{(16 + r)^2 - r^2} = \sqrt{16^2 + 32r} = 4\sqrt{16 + 2r}$$

$$\therefore \quad BD = BS = 48$$

Now, $\Delta ACD \sim \Delta ABS$



$\therefore \quad \frac{AD}{AS} = \frac{CD}{BS}$

$\Rightarrow \quad \frac{4\sqrt{16+2r}}{16+2r} = \frac{r}{48}$

$\Rightarrow \quad \frac{4}{\sqrt{16+2r}} = \frac{r}{48} \Rightarrow r\sqrt{16+2r} = 4 \times 48$

$\Rightarrow \quad r^2(r+8) = 24^2 \times 32$

$\Rightarrow \quad r = 24$

$\therefore \quad$ diameter of wall $= 48$ m.

26. Positive integers x, y, z satisfy xy + z = 160. Compute the smallest possible value of x + yz.

26. (50)

For minimum value of $x + yz$,

$x \ge y \ge z$

$160 = xy + z \ge z^2 + z.$

$\therefore \quad z \le 12,$

Different possibilities for minimum value of $x + yz$ with the Condition $xy + z = 160$

x	y	z	x + yz
53	3	1	56
79	2	2	83
–	–	3	Not possible
26	6	4	50
31	5	5	56

Further Cases will result in a value of higher than 50.

27. We will say that a rearrangement of the letters of a word has no fixed letters if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, H B R A T A is a rearrangement with no fixed letters of B H A R A T. How many distinguishable rearrangements with no fixed letters does B H A R A T have ? (The two As are considered identical.)

27. (84)

BHARAT

The two A's will replace two of the letters B,H,R,T in $^4C_2$ ways.

Let two A's replaces B and H, then

Case-I: R & T are inter changed.

A	A		T		R

Then B and H can be placed in 2 ways.

Case-II: exctly one of R and T goes to the intial position of one of A.

In this case no. of ways = 2×2×2=8

Case-III : Both of R and T goes to initial positions of the two A's.

In this case no. of ways = 2×2=4

$\therefore$ Total no. of required rearrangements

${}^4C_2 \times (2+8+4) = 84$

28. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of ΔABC. Draw tangenets to Ω which are parallel to the sides of ABC. Let r_1, r_2, r_3 be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

28. **(15)**

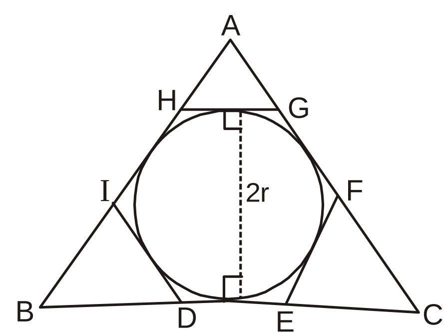
For $\Delta ABC,\ a = 51, b = 52, c = 53$

$$s = \frac{a+b+c}{2} = \frac{51+52+53}{2} = 78$$

$\Delta\sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{78\times 27\times 26\times 25} = 1170$

$\therefore\ r = \frac{\Delta}{s} = 15$



Also $\Delta = \frac{1}{2}\times a\times h_a = rs \Rightarrow \frac{r}{h_a} = \frac{a}{25}$

$\therefore\ HG \parallel BC$

$\therefore\ \Delta AHG \sim \Delta ABC$

$\frac{r_1}{r}$ altitude of ΔAHG through A

$= \frac{h_a - 2r}{h_a}$

$$\Rightarrow r_1 = \frac{r(h_a - 2r)}{h_a} = r(r - \frac{2r}{h_a}) = r(1 - \frac{a}{s})$$

smilarly $r_2 = r(1-\frac{b}{s})$ and $r_3 = r(1-\frac{c}{s})$

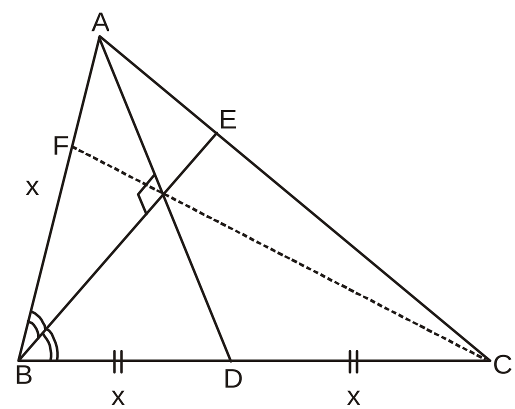
$$\therefore\ r_1 + r_2 + r_3 = r(3 - \frac{a+b+c}{s}) = r(3 - \frac{2s}{s}) = r \Rightarrow 15$$

29. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If AD = 7 and BE = 9, find the integer nearest to the area of triangle ABC.

29. (47)

Let AD and BE meet at G.

Join CG and produce it to meet AB at F.



$\Delta ABG \cong \Delta BG$

$\therefore \ AB = BD = DC = x$ (say)

$\because$ BE is angle bisector.

$\therefore \ \dfrac{CE}{EA} = \dfrac{BC}{BA} = \dfrac{2x}{x} = 2:1$

Now by ceva's theorem,

$\because$ AD, BE and CF are concurrent

$\therefore \ \dfrac{BD}{DC} \times \dfrac{CE}{EA} \times \dfrac{AF}{FB} = 1$

$\Rightarrow \ \dfrac{1}{1} \times \dfrac{2}{1} \times \dfrac{AF}{FB} = 1 \Rightarrow \dfrac{AF}{FB} = 1:2$

$\therefore \ \dfrac{AG}{GD} = 1:1 \quad (\because \ AG = GD)$

and $\dfrac{BG}{GE} = \dfrac{BF}{FA} + \dfrac{BD}{DC} = \dfrac{2}{1} + \dfrac{1}{1} = 3:1$

$\therefore \ AG = \dfrac{AD}{2} = \dfrac{7}{2}$ and $BG = BE \times \dfrac{3}{4} = \dfrac{27}{4}$.

$\therefore \ Ar(\Delta\, ABG) = \dfrac{1}{2} \times \dfrac{7}{2} \times \dfrac{27}{4} = \dfrac{189}{16}$

$\therefore \ Ar(\Delta\, ABD) = 2\ (ar(\Delta\, ABG) = \dfrac{189}{8}$

Hence $ar(\Delta\, ABC) = 2(ar\,(\Delta\, ABD)) = \dfrac{189}{4}$

=47.25

≈ 47 sq. units

30. Let E denote the set of all natural numbers n such that 3 < n < 100 and the set {1, 2, 3, ...,n} can be partitioned in to 3 subsets with equal sums. Find the number of elements of E.

30. **(64)**

We have $\{1,2,3,4,5\} = \{5\} \cup \{1,4\} \cup \{2,3\}$ and $\{1,2,3,4,5,6\} = \{1,6\} \cup \{2,5\} \cup \{3,4\}$

So, 5, 6 $\in$ E.

Now suppose m $\in$ E and {P_1, P_2, P_3} be a partition of {1, 2, ..., m} with 1 $\in P_1$ then $\left(P_1 \cup \{m+3\}\right) - \{1\}$, $P_2 \cup \{m+2\}, P_3 \cup \{m+1, 1\}$ is a partition of {1, 2, ..., m, m + 1, m + 3} with equal sums.

So, whenever m $\in$ E, m + 3 also belongs to E, but 5, 6 $\in$ E

$\therefore \quad 3k-1, 3k \in E \quad \forall \quad k \in \{2,3,....,33\}$

$$\because \; 1+2+3+...+n = \frac{n(n+1)}{2}$$

$$= \frac{(3k+1)(3k+2)}{2}, \text{ for n = 3k + 1 which is not a multiple of 3.}$$

So, $3k+1 \notin E$ for any $k \in I^+$

Thus, E consists of all numbers of the form $3k-1, 3k$ for $k \in \{2,3,....,33\}$.

So, number of elements in E = 64.

PRMO_2019_ANSWER KEY_11-08-2019

1. (04)	**2. (13)**	**3. (13)**	**4. (36)**
5. (10)	**6. (29)**	**7. (51)**	**8. (49)**
9. (14)	**10. (55)**	**11. (6)**	**12. (18)**
13. (10)	**14. (53)**	**15. (45)**	**16. (40)**
17. (30)	**18. (44)**	**19. (13)**	**20. (Bonus)**
21. (17)	**22. (78)**	**23. (55)**	**24. (37)**
25. (48)	**26. (50)**	**27. (84)**	**28. (15)**
29. (47)	**30. (64)**		