RMO 2024-25

(REGIONAL MATHEMATICS OLYMPIAD)

PAPER WITH SOLUTION

Time : 3 hours **November 3, 2024**

Instructions:

- **Calculators (in any form) and protractors aro not allowed.**
- **Rulers and compasses are allowed.**
- **All questions carry equal marks. Maximum marks: 102.**
- **No marks will be awarded for stating an answer without justification.**
- **Answer all the questions.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

[:Q.1] Let n > 1 be a positive integer. Call a rearrangement $a_1, a_2, \dots, a_n$ of 1, 2, ..., n nice if for every k = 2, 3, ... , n, we have that $a_1 + a_2 + \dots + a_k$ is not divisible by k.

(a) If n > 1 is odd, prove that there is no nice rearrangement of 1, 2, ..., n.

(b) If n is even, find a nice rearrangement of 1, 2, ..., n.

[:SOLN] (a) If n is odd, n +1 is even.

$$\therefore a_1 + a_2 + \dots + a_n = 1 + 2 + 3 + \dots + n$$

$$= \frac{n(n+1)}{2} = n \times \frac{n+1}{2}$$

which is divisible by n.

So we cannot have any nice rearrangement of 1, 2, ..., n.

(b) Let n = 2m, $m \in N$.

consider the rearrangement 2, 1, 4, 3, 6, 5, ..., 2m, 2m – 1

If k is even, then

$$a_1 + a_2 + \dots + a_k = (2 + 4 + 6 + \dots \text{to } \frac{k}{2} \text{ terms})$$

$$= 2 \times \frac{\frac{k}{2}\left(\frac{k}{2}+1\right)}{2} + \left(\frac{k}{2}\right)^2$$

$$= \left(\frac{k}{2}\right)(k+1)$$

$\because k+1$ is coprime to k and $\frac{k}{2} \in N$

so k does not divide $a_1 + a_2 + \dots + a_k$.

If k is odd, then k = 2p + 1 for some $p \in N$

$$\therefore a_1 + a_2 + \dots + a_k$$

= (2 + 4 + 6 + ... to p + 1 terms) + (1 + 3 + 5 + ... to p terms)

$$= 2 \times \frac{(p+1)(p+2)}{2} + p^2$$

$= 2p^2 + 3p + 2 = (2p + 1)(p + 1) + 1 = k(p + 1) + 1$

which is not divisible by k.

So the rearrangement 2, 1, 4, 3, 6, 5, ..., 2m, 2m – 1 is nice.

[:Q.2] For a positive integer n, let R(n) be the sum of the remainders when n is divided by 1, 2,..., n. For example, R(4) = 0+0+1+0 = 1, R(7) = 0+1+1+3+2+1+0 = 8. Find all positive integers n such that R(n) = n –1.

[:SOLN] **Case I :** n = 2m, $m \in N$.

Then the remainder when n is divided by k

= n – k for each $k \in \{m+1, m+2, ..., 2m\}$

$$\therefore R(n) \geq (n-(m+1)) + (n-(m+2)) + ... + (n-2m)$$

$$\Rightarrow n-1 \geq (m-1) + (m-2) + ... + 0$$

$$\Rightarrow 2m-1 \geq \frac{m(m-1)}{2}$$

$$\Rightarrow 4m-2 \geq m^2 - m$$

$$\Rightarrow m^2 - 5m + 2 \leq 0$$

$$\Rightarrow m \in \{1, 2, 3, 4\}$$

$$\therefore n \in \{2, 4, 6, 8\}$$

but, R(2) = 0 + 0 = 0

R(4) = 0 + 0 + 1 + 0 = 1

R(6) = 0 + 0 + 0 + 2 + 1 + 0 = 3

R(8) = 0 + 0 + 2 + 0 + 3 + 2 + 1 + 0 = 8

So none of these satisfies the condition.

Case II : $n = 2m-1, m \in N$

Then the remainder when n is divided by k = n – k for each

$k \in \{m, m+1, m+2, ..., 2m-1\}$

$$\therefore R(n) \geq (n-m) + (n-(m+1)) + ... + (n-(2m-1))$$

$$\Rightarrow n-1 \geq (m-1) + (m-2) + ... + 0$$

$$\Rightarrow 2m-2 \geq \frac{m(m-1)}{2} \Rightarrow m^2 - 5m + 4 \leq 0$$

$$\Rightarrow (m-1)(m-4) \leq 0 \Rightarrow m \in \{1, 2, 3, 4\}$$

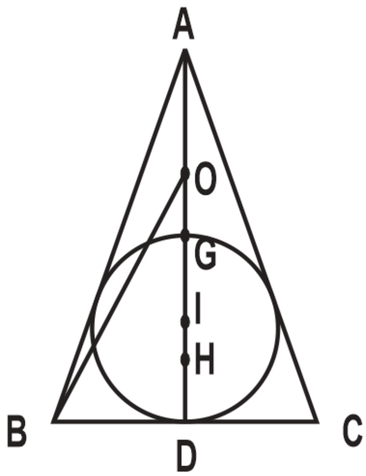
$$\therefore n \in \{1, 3, 5, 7\}$$

but R(1) = 0, R(3) = 1, R(5) = 4, R(7) = 8.

So only n = 1 and n = 5 satisfies the condition.

[:Q.3] Let ABC be an acute triangle with AB = AC. Let D be the point on BC such that AD is perpendicular to BC. Let O, H, G be the circumcentre, orthocentre and centroid of triangle ABC respectively. Suppose that 2·OD = 23·HD. Prove that G lies on the incircle of triangle ABC.

[:SOLN]



$\because AB = AC$

$\therefore O, G, I, H$ all lie on the altitude AD.

Let HD = x.

Then OD = $\frac{23}{2}HD = \frac{23}{2}x$.

Now AH = 2 OD = 23 HD = 23x

$\therefore AD = AH + HD = 24x$

$\therefore AO = AD - OD = 24x - \frac{23}{2}x = \frac{25}{2}x = OB$

$\therefore BD = \sqrt{OB^2 - OD^2} = \sqrt{\left(\frac{25}{2}x\right)^2 - \left(\frac{23}{2}x\right)^2} = 2\sqrt{6}x = DC$

$AB = \sqrt{AD^2 + BD^2} = \sqrt{(24x)^2 + \left(\sqrt{24}x\right)^2} = 10\sqrt{6}x$

$\therefore$ inradius, r = $\dfrac{\Delta}{s} = \dfrac{\frac{1}{2}\times BC \times AD}{\frac{1}{2}(AB + AC + BC)}$

$$= \frac{\frac{1}{2}\times 2(2\sqrt{6}x)\times 24x}{\frac{1}{2}\left(10\sqrt{6}x + 10\sqrt{6}x + 4\sqrt{6}x\right)}$$

$$= \frac{4\sqrt{6}\times 24x^2}{24\sqrt{6}x} = 4x$$

Now GD = $\frac{1}{3}(AD) = 8x = 2r$

Hence G lies on the incircle.

[:Q.4] Let a_1, a_2, a_3, a_4 be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1.$ Show that there exist i, j with $1 \le i < j \le 4,$ such that $(a_i - a_j)^2 \le \frac{1}{5}.$

[:SOLN] We have

$$\sum_{1\le i<j\le 4}(a_i - a_j)^2 = 3\left(\sum_{i=1}^{4} a_i^{\,2}\right) - 2\sum_{1\le i<j\le 4} a_i a_j$$

$$= 4\left(\sum_{i=1}^{4} a_i^{\,2}\right) - \left(\sum_{i=1}^{4} a_i\right)^2$$

$$= 4 - \left(\sum_{i=1}^{4} a_i\right)^2 \le 4$$

Without loss of generality, we can assume that $a_1 \le a_2 \le a_3 \le a_4$

Let $a_2 - a_1 = x$, $a_3 - a_2 = y$ and $a_4 - a_3 = z$.

Then $\sum_{1\le i<j\le 4}(a_i - a_j)^2 = x^2 + y^2 + z^2 + (x+y)^2 + (y+z)^2 + (z+x)^2 + (x+y+z)^2$

If each of $x, y, z > \frac{1}{\sqrt{5}}$, then

$$\sum_{1\le i<j\le 4}(a_i-a_j)^2 > \left(\frac{1}{\sqrt{5}}\right)^2+\left(\frac{1}{\sqrt{5}}\right)^2+\left(\frac{1}{\sqrt{5}}\right)^2+\left(\frac{2}{\sqrt{5}}\right)^2+\left(\frac{2}{\sqrt{5}}\right)^2+\left(\frac{3}{\sqrt{5}}\right)^2=4$$

which is a contradiction.

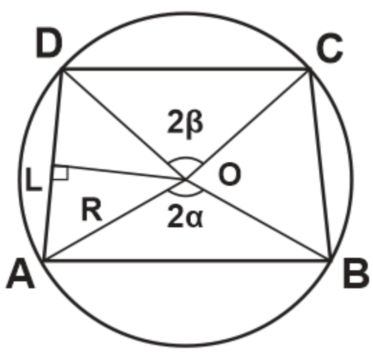
Hence at least one of $x,y,z \le \frac{1}{\sqrt{5}}$ and so there exists i, j with $1\le i<j\le 4$

such that $(a_i-a_j)^2 \le \left(\frac{1}{\sqrt{5}}\right)^2=\frac{1}{5}$.

[:Q.5] Let ABCD be a cyclic quadrilateral such that AB is parallel to CD. Let O be the circumcentre of ABCD, and L be the point on AD such that OL is perpendicular to AD. Prove that

$$OB\cdot(AB+CD)=OL\cdot(AC+BD).$$

[:SOLN]



$\because AB \,||\, CD$

$\therefore \angle ABC+\angle BCD=180^\circ$

$\because ABCD$ is cyclic quadrilateral

$\therefore \angle BAD+\angle BCD=180^\circ$

So $\angle BAD = \angle ABC$

Hence ABCD is an isosceles trapezium with AD = BC.

Let OA = OB = OC = OD = R, $\angle AOB=2\alpha$ and $\angle COD=2\beta$

$\therefore AB=2R\sin\alpha$, CD = 2R sin β

$\Delta AOD\cong\Delta BOC$

$$\therefore \angle AOD=\angle BOC=\frac{360^\circ-(2\alpha+2\beta)}{2}$$

$$= 180^\circ - (\alpha + \beta)$$

$$\therefore OL = R\cos\left(\frac{180^\circ - (\alpha+\beta)}{2}\right) = R\sin\frac{\alpha+\beta}{2}$$

$$\angle AOC = \angle AOD + \angle DOC = 180^\circ - (\alpha+\beta) + 2\beta$$

$$= 180^\circ - (\alpha - \beta) = \angle BOD$$

$$\therefore AC = BD = 2R\sin\left(\frac{\angle AOC}{2}\right) = 2R\sin\left(90^\circ - \frac{\alpha-\beta}{2}\right)$$

$$= 2R\cos\frac{\alpha-\beta}{2}$$

So $OB.(AB+CD) = R(2R\sin\alpha + 2R\sin\beta)$

$$= 2R^2.2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$= 2\left(R\sin\frac{\alpha+\beta}{2}\right)\left(2R\cos\frac{\alpha-\beta}{2}\right)$$

= 2·OL·AC

= OL(AC + BD) $(\because AC = BD)$

[:Q.6] Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \ldots, a_k$ of integers an n-chain if $1 = a_1 < a_2 < \ldots < a_k = n$, and a_i divides a_{i+1} for all $i, 1 \leq i \leq k-1$. Let f(n) be the number of n-chains where $n \geq 2$. For example, f (4) = 2 corresponding to the 4-chains {1,4} and {1,2,4). Prove that f $(2^m \cdot 3) = 2^{m-1}(m + 2)$ for every positive integer m.

[:SOLN] Let $n = 2^m.3$

The divisors of $2^m.3$ are

$1, 2, 2^2, \ldots, 2^m, 3, 2.3, 2^2.3, \ldots, 2^m.3$

Consider an n-chain containing $2^p.3$, where p is the least number such that the n-chain contains $2^p.3$.

Case I : p < m.

Then it will be a subset of $\left\{1, 2, 2^2, \ldots, 2^p, 2^p.3, 2^{p+1}.3, \ldots, 2^m.3\right\}$

where it must contain 1, $2^p.3$ and $2^m.3$

So number of such n-chains = $2^{(m+2)-3} = 2^{m-1}$

Now number of possible values of p = m $\quad (\because p \in \{0,1,2,...,m-1\})$

So the number of such n-chains = $2^{m-1}.m$

Case II : p = m.

Then any such n-chain is a subset of $\{1, 2, 2^2, \ldots, 2^m, 2^m.3\}$,

where it must contain 1& $2^m.3$

So number of such n-chains = 2^m

So $f(2^m.3) = 2^{m-1}.m + 2^m$

$= 2^{m-1}(m + 2)$.