November 16, 2025

RMO - 2025
QUESTIONS & SOLUTIONS

Time : 3 hours

Instructions :

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. **Maximum marks : 102.**
- Answer to each question should start on a new page. Clearly indicate the question number.

01. (a) Let $n \geq 3$ be an integer. Find a configuration of n lines in the plane which has exactly

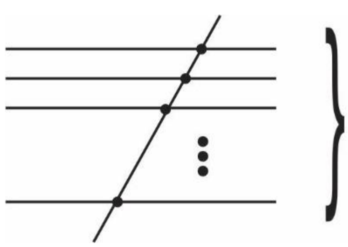
(i) $n-1$ distinct points of intersection;

(ii) n distinct points of intersection;

(b) Give configurations of n lines that have exactly n + 1 distinct points of intersection for (i) n = 8 and (ii) n = 9.

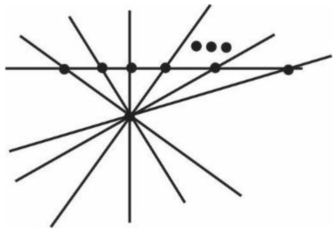
Soln. **[A]**

(i) Considering 'n – 1' parallel lines and a line not parallel to it.



As evident from adjacent diagram, there are n – 1 distinct points of intersections.

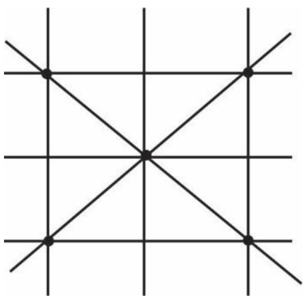
(ii) Considering n – 1 concurrent lines and a line drawn not through the point of concurrency and non-parallel to each of the n – 1 lines the lines to meet all n – 1 lines.



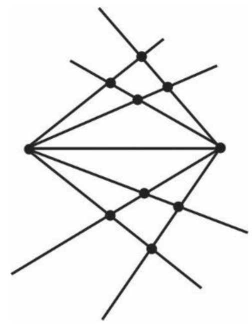
so there are n points of intersection in total

[B]

(i)



(ii)



02. Let a, b, c be distinct nonzero real numbers satisfying

$$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}$$

Determine the value of $\left|a^2b+b^2c+c^2a\right|$.

Soln. $a+\frac{2}{b}=b+\frac{2}{c}$

$\Rightarrow \quad a-b=\frac{2}{c}-\frac{2}{b}=\frac{2(b-c)}{bc}$

similarly, $b-c=\frac{2(c-a)}{ca}$

and $c-a=\frac{2(a-b)}{ab}$

multiplying, $(a-b)(b-c)(c-a)=\frac{8(a-b)(b-c)(c-a)}{a^2b^2c^2}$

since a, b, c are distinct

$\therefore \quad 1=\frac{8}{a^2b^2c^2} \Rightarrow abc=\pm 2\sqrt{2}$

Now $a+\frac{2}{b}=b+\frac{2}{c}$

$\Rightarrow \quad (ab+2)c=(bc+2)b$

$\Rightarrow \quad abc+2c=b^2c+2b$

$\Rightarrow \quad b^2c=abc+2c-2b$

similarly, $a^2b=abc+2b-2a$

and $c^2a=abc+2a-2b$

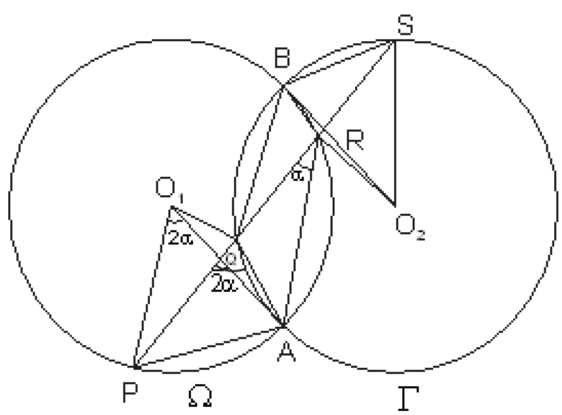
Adding, we get $a^2b+b^2c+c^2a=3abc$

$\therefore \quad \left|a^2b+b^2c+c^2a\right|=|3abc|=6\sqrt{2}$

03. Let Ω and Γ be circles centred at O_1, O_2 respectively. Suppose that they intersect in distinct points A, B. Suppose O_1 is outside Γ and O_2 is outside Ω. Let ℓ be a line not passing through A and B that intersects Ω at P, R and Γ at Q, S so that P, Q, R, S lie on the line in this order. Furthermore, the points O_1, B lie on one side of ℓ and the points O_2, A lie on the other side of ℓ. Given that the points A, P, Q, O_1 are concyclic and B, R, S, O_2 are concyclic as well, prove that AQ = BR.

Soln. Join QB, AR, O_1A, and O_2 B.

PAQ O_1 is cyclic quad



$\therefore \quad \angle PQA=\angle PO_1A$

$=2\angle PRA$

$=2\alpha$ (say)

So $\angle QAR=\angle PQA-\angle QRA$

$= 2\alpha - \alpha = \alpha$

$= \angle QRA$

$\therefore$ in isosceles Δ AQR,

$QR = AQ$ .... (1)

Similarly, we can show that

$QR = BR$ ..... (2)

From (1) & (2),

$AQ = BR$ proved.

04. Prove that there do not exist positive rational numbers x and y such that

$$x + y + \frac{1}{x} + \frac{1}{y} = 2025$$

Soln. Let $x = \frac{p}{q}$ and $y = \frac{r}{s}$, where p, q, r, s are positive integers such that

gcd(p, q) = 1 and gcd(r, s) = 1,

$\therefore \quad x + y + \frac{1}{x} + \frac{1}{y} = 2025$

$\Rightarrow \quad \frac{p}{q} + \frac{r}{s} + \frac{q}{p} + \frac{s}{r} = 2025$

$\Rightarrow \quad \frac{p^2 + q^2}{pq} + \frac{r^2 + s^2}{rs} = 2025$

$\Rightarrow \quad rs\left(p^2 + q^2\right) + pq\left(r^2 + s^2\right) = 2025\ pqrs$ ....(1)

$\Rightarrow \quad rs\left(p^2 + q^2\right) = pq\left(2025\ rs - r^2 - s^2\right)$

$\therefore$ pq | rs(p^2 + q^2)

But gcd (pq, p^2 + q^2) = 1

$\therefore$ pq | rs similarly rs | pq $\therefore$ pq = rs

So we may assume that p = ab, q = cd, r = ad and s = bc

Where a, b, c, d are pairwise coprime positive integers.

So from (1),

abcd (a^2b^2 + c^2d^2) + abcd(a^2d^2 + b^2c^2) = 2025 $a^2b^2c^2d^2$

$\Rightarrow \quad a^2b^2 + c^2d^2 + a^2d^2 + b^2c^2 = 2025$ abcd

$\Rightarrow \quad (a^2 + c^2)(b^2 + d^2) = 2025$ abcd

$= 3^4 \times 5^2$ abcd

So one of $a^2 + c^2$ and $b^2 + d^2$ must be divisible by 9

Let us assume that $9 \mid a^2 + c^2$.

but $a^2 \equiv 0, 1, 4, 7 \bmod 9$

and $c^2 \equiv 0, 1, 4, 7 \bmod 9$

so $a^2 + c^2 \equiv 0 \bmod 9$

iff $\quad a^2 \equiv 0 \bmod 9$ and $c^2 \equiv 0 \bmod 9$

$\Rightarrow \quad 3 \mid a$ and $3 \mid c$

Which contradicts that a and c are coprime.

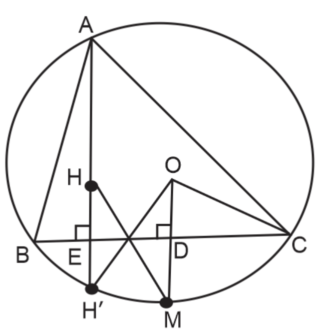
So there do not exists rational numbers x and y

satisfying $x + y + \frac{1}{x} + \frac{1}{y} = 2025$.

05. Let ABC be an-acute-angled triangle with AB < AC, orthocenter H and circumcircle Ω. Let M be the midpoint of minor arc BC of Ω. Suppose that MH is equal to the radius of Ω. Prove that $\angle BAC = 60^o$.

Soln. Let O be the circumcenter of ΔABC and H′ be the image of H about side BC. So H′ also lies on the circumcircle Since M is mid print if arc BC So $OM \perp BC$

But $HH' \perp BC$.



$\therefore \qquad HH' \parallel OM$

Now $OH' = HM =$ radius of circumcircle so $HH'MO$ is an isosceles trapezium with $HH' \| OM$. But BC is perpendicular bisector of HH^1 So BC is also perpendicular bisector of OM.

$\therefore \quad OD = \frac{OM}{2} = \frac{OC}{2} \qquad (\because OM = OC)$

$\therefore \quad$ in ΔODC

$\cos(\angle DOC) = \frac{OD}{OC} = \frac{1}{2}$

$\Rightarrow \quad \angle DOC = 60^o$

$\therefore \quad \angle BAC = \frac{1}{2}\angle BOC = \angle DOC = 60^o$

06. Let p(x) be a nonconstant polynomial with integer coefficients, and let $n \geq 2$ be an integer such that no term of the sequence

p(0), p(p(0)), p(p(p(0))),…….

is divisible by n. Show that there exist integers a, b such that $0 \leq a < b \leq n-1$ and n divides p(b) – p(a).

Soln. Let us denote $\underbrace{p(p(p(.......p(0).....)))}_{k\ times}$ by a_k for each k.

Then the sequence is a_1, a_2, a_3, ………

Where $a_{k+1} = p(a_k)$ for each k and $a_1 = p(0)$

Now consider the remainders when the terms of the sequence a_1, a_2, a_3, ……, a_n are divided by n. Since there are only n – 1 possible remainders 1, 2, 3, …….., n – 1, so by PHP, there must be at least two terms giving the same remainder. Let k and l be two smallest numbers with $k \neq l$ such that $a_{k+1} \equiv a_{l+1}$ mod n

$\Rightarrow \quad p(a_k) \equiv p(a_l)$ mod n

Let $a_k \equiv nq_1 + a$ and $a_l \equiv nq_2 + b$, q_1, $q_2 \in$ I and a, b $\in$ {0, 1, 2, ……., n – 1}.

If a = b, then $a_k \equiv a_l$ mod n which contradicts the assumption.

So $a \neq b$. Without loss of generality

We may assume that a < b.

Now $p(a_k) - p(a) \equiv a_k - a$

$\equiv na_1$

$\equiv 0 \bmod n$

$\therefore \quad p(a_k) \equiv p(a) \bmod n$

Similarly $p(a_l) \equiv p(b) \bmod n$

$\therefore \quad p(b) - p(a) \equiv$

$\equiv p(a_l) - p(a_k)$

$\equiv 0 \bmod n$

i.e. $n \mid p(a) - p(b)$